

Math 114, solutions to Assignment 12

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These are the solutions to the twelfth homework assignment.

1 Section 10.3 # 24

We can write the equation as

$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{ye^y}$$

and multiplying through by $ye^y dx$ gives

$$ye^y dy = x\sqrt{x^2+1} dx.$$

Integrating both sides (the left side by parts, the right side by making the substitution $u = x^2 + 1$) gives

$$(y-1)e^y = \frac{1}{3}(x^2+1)^{3/2} + C.$$

To plot this, use the `implicitplot` command in Maple. The Maple code necessary to plot a single such solution curve is something like:

```
with(plots)
implicitplot((y-1)*exp(y)=1/3*(x^2+1)^(3/2)-C, x=-5..5, y=-5..5);
```

where C is replaced with some actual constant. The principal things to notice are that the curves have two branches and are symmetric around the y -axis, and for very large positive or negative C do not get close to the origin. When C is small in absolute value, the curves get close to the origin; for $C < -4/3$ the curves pass to the left and right of the origin, and for $C > -4/3$ they pass above and below the origin. The solution curve for $C = 4/3$ is a special case which passes through the origin; the plot of

$$(y-1)e^y = \frac{(x^2+1)^{3/2}}{3} - \frac{4}{3}$$

looks like the letter “X” there. This is a singularity, a point where the differential equation can’t be solved; notice that not only is the function

$$\frac{x\sqrt{x^2+1}}{ye^y}$$

not *defined* at $(x, y) = (0, 0)$, it doesn’t even have a limit there. This explains the singularity.

2 Section 18.3 # 10.

We want to solve the differential equation

$$mx'' + kx = F_0 \cos \omega t.$$

The general solution to the corresponding homogeneous equation

$$mx'' + kx = 0$$

is given by $x = c_1 \cos \omega t + c_2 \sin \omega t$, where $\omega = \sqrt{k/m}$. Now, we need to find a particular solution to our original equation. We cannot assume it has the form $\cos \omega t$ or $\sin \omega t$ since these are solutions to the homogeneous equation. So we assume that the equation has a solution of the form

$$x = At \cos \omega t + Bt \sin \omega t$$

and differentiating twice gives

$$x'' = A(-2\omega \sin \omega t - \omega^2 t \cos \omega t) + B(2\omega \cos \omega t - \omega^2 t \sin \omega t).$$

Thus, we have

$$mx'' + kx = (-2Am\omega) \sin \omega t + (2Bm\omega) \cos \omega t + B(-m\omega^2 + k)t \sin \omega t + A(-m\omega^2 + k)t \cos \omega t.$$

Recalling $k = m\omega^2$, this is just

$$-2Am\omega \sin \omega t + 2Bm\omega \cos \omega t.$$

Thus, we must have

$$-2Am\omega \sin \omega t + 2Bm\omega \cos \omega t. = F_0 \cos \omega t$$

from which $B = F_0/(2m\omega)$, $A = 0$, and we have the general solution

$$x = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

as desired.

3 Qualitative differential equations problem

(a) Notice that $0 \leq y' \leq 1$ for all positive t and y . First consider when t is positive. We have $y(0) = 1$ and $y'(t) < 1$ for all $t > 0$. Thus

$$y(t) = 1 + \int_0^t y'(u) du \leq 1 + \int_0^t 1 du = 1 + t$$

as desired.

Now, when t is negative, notice that $y'($

Alternatively, we have $y' = y/(y+t)$, and by the quotient rule

$$y'' = \frac{(y+t)y' - y(y+t)'}{(y+t)^2}.$$

Recalling $y' = y/(y+t)$, we have

$$y'' = \frac{(y+t)\frac{y}{y+t} - y\left(\frac{y}{y+t} + 1\right)}{(y+t)^3}$$

Simplifying gives

$$y'' = \frac{-y^2}{(y+t)^3}.$$

(b) We know $y \leq 1+t$ for large t , from part (a). Thus $y \leq 2t$ for large t , since $1+t \leq 2t$ for large t . Now, if $y < 2t$, then we have $y' = y/(y+t) < 2/3$; since y' remains below $2/3$, we eventually have $y < (2/3)t$. (This isn't strictly correct; if $y' \rightarrow 2/3$ as $t \rightarrow \infty$ then this might not be true. But we didn't ask for a formal proof.)

(c) For large enough t , $y \leq 2/3t$. Now, if $y < \frac{2}{3}t$, then we have $y' = \frac{y}{y+t} < \frac{2}{5}$; since y' remains below $2/5$ for large t , we eventually have $y < \frac{2}{5}t$. (The same caveat as above applies.) Similarly, we can show that for large t we have $y < \frac{2}{7}t, y < \frac{2}{9}t$, and so on; repeating enough times, $y < ct$ for any positive constant c .

(d) Since $y(t) > 1$ for all positive t , we have $y' > 1/(1+t)$. By comparison, then,

$$y(t) > 1 + \int_0^t \frac{1}{1+u} du$$

where the constant 1 comes from the initial condition. Thus $y(t) > 1 + \log(1+t)$, and this lower bound for $y(t)$, increases without bound, so $y(t)$ increases without bound as well.

(A lot of people said that since $\lim_{t \rightarrow \infty} y'(t) = 0$, then $\lim_{t \rightarrow \infty} y(t)$ exists. There are two problems with this. First, those people for the most part didn't show that $y'(t)$ actually approaches zero; I am led to assume that they made such a guess from a picture, but such pictures can be misleading, and in any case, if you make such a guess you should at least give some reasoning. Second, it's not true. Let $f(t) = \log t$, so then $f'(t) = \frac{1}{t}$; we have $\lim_{t \rightarrow \infty} f'(t) = 0$ but $\lim_{t \rightarrow \infty} f(t) = \infty$. This is exactly analogous to the divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which should be familiar.)

4 An exact solution to the differential equation

We had the differential equation

$$\frac{dy}{dt} = \frac{y}{y+t}$$

which cannot be solved by the techniques that we have covered in class. However, we can take the reciprocal of both sides to get

$$\frac{dt}{dy} = \frac{y+t}{t}$$

which is in fact a linear equation with independent variable y and dependent variable t , as we can see if we rewrite it in the form

$$\frac{dt}{dy} - \frac{1}{t}y = 1$$

This is a linear equation with solution $t = y \log y$, as you can verify.

This enables us to solve (b), (c), and (d) quite easily. To show that $y < ct$ for large t , it suffices to show that $t > Cy$ for large y , where $C = 1/c$. But clearly for $y > e^C$, we have $t = y \log y > Cy$. This solves (c); (b) is the special case of (c) with $c = 2/3$ and thus $C = 3/2$. For (d), note that we can make y as large as we like just by making t large enough; for any positive real constant k , if $t > k \log k$ then $y > k$.

Another way to get the same exact solution is to let $u = y/t$, so $y = ut$. Differentiating this, by the product rule we have

$$\frac{dy}{dt} = \frac{du}{dt}t + u$$

and so our original differential equation becomes

$$t \frac{du}{dt} + u = \frac{ut}{ut+t};$$

which can be rewritten as

$$t \frac{du}{dt} = \frac{u}{u+1} - u$$

and the equation is seen to be separable. In particular, we have

$$\frac{-(1+u)}{u^2} du = \frac{dt}{t}.$$

Integrating both sides gives

$$\frac{1}{u} - \log u = \log t + C.$$

Recalling that $u = y/t$, we get

$$\frac{t}{y} - \log y/t = \log t + C$$

and adding $\log y/t$ to both sides and simplifying,

$$\frac{t}{y} = \log y + C.$$

Substituting the initial condition gives $C = 0$; thus $\frac{t}{y} = \log y$, or $t = y \log y$.