Math 114, solutions to Assignment 2

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These are the solutions to the second homework assignment. The plain text represents solutions to the problems; the *italic text* represents my comments on the solution.

1 The valley problem

We begin by determining the position of the sun in the sky. The sun casts its shadow exactly in the northeast direction, which corresponds to the line $y = x$. The southeastern hillside, as viewed from above, is entirely south and east of the line $y = x/2$; the northwestern hillside is north and west of the line $y = 2x$. Thus the shadow shines onto the floor of the valley.

The position of the sun in the sky is thus exactly in the southwest, in the direction $(-1, -1, c)$ for some constant c to be determined. In a coordinate system with units in feet, the top of the tree is at the point $(0, 0, 100)$ and the end of its shadow is at $(100, 100, 0)$; the direction of the vector between these points is the direction of the sun. This vector is $(-100, -100, 100)$, which has the same direction as $\vec{s} = (-1, -1, 1)$; thus $c = 1$.

A common mistake I saw here was to assume that the direction in which the sun shines is given by $(100, 100, 0)$*, the vector corresponding to the shadow; this doesn't make sense physically, because this would mean the sun's rays were horizontal. Also, it is not necessary to convert the feet here to miles, since the ratio between the tree's length and the shadow's length is what matters; however, it's not harmful to do so. Some people gave the vector of the sun as something like* $(100/5280, 100/5280, -100/5280)$*, though, which created something of an arithmetical mess. A less common mistake was to compute the normal vector of the plane determined by the origin, the top of the tree, and the end of the shadow.*

The northwestern hillside contains the origin and the points $(1, 2, 0)$ and $(0, 1, 1/4)$; a normal vector to the hillside can be obtained by taking the cross product of the vectors corresponding to these two points, giving $\vec{n}_1 = (1/2, -1/4, 1)$. *I saw other cross products taken; for example, some people took* $(0, 1, 1/4) \times (1, 2, 0) = (-1/2, 1/4, -1)$*. Since the normal vector is determined only up to sign, this is okay, although one needs to take this into account when later computing angles. Also, a lot of people gave cross products where the sign of one* *component was reversed; be careful of sign errors!*

Similarly, the southeastern hillside contains the origin and the points $(2, 1, 0)$ and $(3/2, 1/2, 1/10)$; the cross product of these is the normal vector to the southeastern hillside, $\vec{n}_2 = (1/10, -1/5, -1/2)$.

We can compute the angle θ_2 between $\vec{s}$ and $\vec{n}_2$, which quantifies the sun exposure on the first (northwestern) hillside; we have

$$\begin{aligned}
\cos\theta_2 &= \frac{\vec{s}\cdot\vec{n}_2}{|\vec{s}|\,|\vec{n}_2|} \\
\cos\theta_2 &= \frac{(-1)\left(\frac{-1}{10}\right) + (-1)\left(\frac{-1}{5}\right) + (1)\left(\frac{-1}{2}\right)}{\sqrt{3}\sqrt{\frac{3}{10}}} \\
\cos\theta_2 &= \frac{-2\sqrt{10}}{15} \\
\theta_2 &= \cos^{-1}\left(\frac{-2\sqrt{10}}{15}\right) = 114.9^o
\end{aligned}$$

It's preferable to not use decimals in this calculation until the very end, because otherwise one could introduce rounding error. I don't think this caused anyone to get the wrong answer, but it's something worth worrying about. Also, some people gave the angle in radians; this is technically correct, but most people think in degrees so it makes more sense to state it in degrees. You only need to use radians when you're differentiating or integrating.

However, the angle between the normal vector and the hillside is an acute angle, so we replace θ_2 with its supplement, $\cos^{-1}\left(2\sqrt{10}/15\right) = 65.1^o$.

A similar calculation of the angle between $\vec{s}$ and $\vec{n}_1$ gives $\theta_1 = \cos^{-1}(1/\sqrt{7}) = 67.7^o$. *You might show this computation.*

The angle between the normal to the southeastern plane and the sun's rays is *smaller* than the angle between the normal to the northwestern plane and the sun's rays. This means that the southeastern plane is more nearly perpendicular to the sun's rays, and is therefore more exposed to the sun.

What could you do if you didn't have a calculator? We have $\theta_2 = \cos^1 2\sqrt{10}/15$, and $\theta_1 = \cos^{-1} 1/\sqrt{7}$. Whichever one of those arguments is larger will have the smaller inverse cosine. You can see that $2\sqrt{10}/15 > 1/\sqrt{7}$ by squaring them, since $40/225 > 1/7$. So θ_2 is the smaller angle.

2 Section 13.5, Problem 74

A general note on this problem: A lot of people just said, of each set of planes, "these are the intercepts, this is the normal vector", and so on. While these things are not wrong, what was desired here was a sort of explanation of how the planes "fit together" in space.

(a). Each of these planes has a normal vector $(1, 1, 1)$. Thus, the planes are all parallel. In addition, each plane has x, y, and z intercepts equidistant from the origin. *Some people, in describing this family, referred to the plane as an "equilateral triangle". THIS IS NOT CORRECT. While the three intercepts do determine an equilateral triangle, the plane stretches infinitely far in all directions. Also, some people stated that the angle between this plane and the coordinate axes is 45 degrees. In fact, it is the angle between* $(1, 0, 0)$ *and* $(1, 1, 1)$, *which can be calculated to be* $\cos^{-1} 1/\sqrt{3}$, *which is about 54.7 degrees.*

(b). This is the family of planes passing through $(1, 0, 0)$ and $(0, 1, 0)$, as can be seen by substituting these points into the generic equation. As c varies, we see all such planes except the xy-plane itself; this plane has equation $z = 0$ which is not a scalar multiple of any equation of the form $x + y + cz = 1$.

(c). These planes are the planes tangent to a cylinder with central axis the x-axis and radius 1. We know that the planes are parallel to the x-axis because the coefficients of x in their equations are zero. The distance from these planes to the origin is given by a formula in Section 13.5, which tells us that the distance from the plane $ax + by + cz + d = 0$ to the origin is

$$D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}};$$

letting $d = -1, a = 0, b = \cos\theta, c = \sin\theta$ gives $D = 1$. We can see from geometry that any such plane is tangent to the cylinder. *Note that the* θ *here is a parameter like the* c *in parts (a) and (b); in particular it is not the* θ *of polar coordinates. It is, in fact, the angle between the* y*-axis and the normal vector to the plane.*