Math 114, solutions to Assignment 3

Isabel Lugo

September 28, 2007

These are the solutions to the third homework assignment. The plain text represents solutions to the problems; the *italic text* represents my comments on those solutions.

1 13.7, Problem 52

We are asked to write the equation $x^2 + y^2 + z^2 + 2z = 0$ (a) in cylindrical coordinates and (b) in spherical coordinates.

To convert to cylindrical coordinates, we note that $x^2+y^2 = r^2$; thus we have $r^2+z^2+2z = 0$. Completing the square, this can be written as $r^2+(z+1)^2 = 1$. *The step of completing the square is not, strictly speaking, necessary, but the equation looks nicer if this is done.*

To convert to spherical coordinates, we note that $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos\phi$; thus we have $\rho^2 + 2\rho\cos\phi = 0$. We can cancel a factor of ρ and rearrange to get $\rho = -2\cos\phi$.

2 The cone problem

(a). Let $\vec{r}(t) = \left\langle t^2 - 1, 2t, t^2 + 1\right\rangle$. Then we have $\vec{r}(2) = \langle 3, 4, 5\rangle$. We take $\vec{r}'(t) = \langle 2t, 2, 2t\rangle$, and $\vec{r}'(2) = \langle 4, 2, 4\rangle$. A vector equation for the line in question is thus

$$\vec{v}(t) = \vec{r}(2) + \vec{r}'(2) = \langle 3, 4, 5\rangle + t\,\langle 4, 2, 4\rangle$$

and thus we have the parametric equations

$$x = 3 + 4t, y = 4 + 2t, z = 5 + 4t.$$

Note that to get the direction of the line you must take $\vec{r}'(2)$, that is, you must substitute $t = 2$ into the expression $\langle 2t, 2, 2t\rangle$. Some people attempted other substitutions.

(b). We begin by finding three points on the curve; we have $\vec{r}(-1) = \langle 0, -2, 2\rangle, \vec{r}(0) = \langle -1, 0, 1\rangle; \vec{r}(1) = \langle 0, 2, 2\rangle$. The plane passes through these three points; thus it contains the vectors

$$\vec{r}(0) - \vec{r}(-1) = \langle -1, 2, -1\rangle, \vec{r}(0) - \vec{r}(1) = \langle -1, -2, -1\rangle.$$

The normal vector to the plane is then given by

$$\vec{n} = \langle -1, 2, -1\rangle \times \langle -1, -2, -1\rangle = -4\hat{\imath} + 4\hat{k}.$$

The equation of the plane is therefore

$$\langle -4, 0, 4\rangle \cdot \langle x, y, z\rangle = \langle -4, 0, 4\rangle \cdot \langle -1, 0, 1\rangle$$

or, after simplifcation, $-x + z = 2$. To check that the curve lies on the plane, we can simply note that $-(t^2 - 1) + (t^2 + 1) = 2$.

(c). We can observe that $(t^2 - 1)^2 + (2t)^2 = (t^2 + 1)^2$, and so the curve lies on the cone $x^2 + y^2 = z^2$.

However, if you don't see this immediately, there are three more systematic ways to get this solution – based on the equation of a cone in spherical, cylindrical, or Cartesian coordinates.

The equation of a cone with vertex at the origin and axis the z-axis, in cylindrical coordinates, is $\phi = c$ for some constant c. But we have from the equations for converting between Cartesian and spherical coordinates that $z = \rho \cos \phi$. So if our curve lies on such a cone, then we will have z/ρ a constant.

We can compute:

$$\frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{t^2 + 1}{\sqrt{2}(t^2 + 1)} = \frac{1}{\sqrt{2}}$$

and so $\frac{z}{\rho}$ is in fact a constant, namely $\pi/4 = \cos^{-1} 1/\sqrt{2}$. Thus the curve lies on a cone.

Similarly, in cylindrical coordinates such a cone has equation $r = kz$ for some constant k; we have $r = \sqrt{x^2 + y^2}$. In fact, for our curve,

$$r = \sqrt{x^2 + y^2} = \sqrt{(t^2 - 1)^2 + (2t)^2} = t^2 + 1 = z$$

and so our cone has equation $r = z$ in cylindrical coordinates.

Finally, if we do the problem in rectangular coordinates, such a cone has equation $x^2 + y^2 = kz^2$. We can find k satisfying this by substituting $x = t^2 - 1, y = 2t, z = t^2 + 1$; we get $(t^2 - 1)^2 + 2(2t)^2 = k(t^2 + 1)^2$, which has a solution, $k = 1$. So the equation of our cone is $x^2 + y^2 = z^2$.

Note that it does not suffice to, for example, say that a cone has equation $x^2 + y^2 = kz^2$ and plug in a single value of t to determine k. All this does is say that if the curve lies on a cone whose vertex is the x-axis, then it lies on the particular cone you've picked out. We don't know from the statement of the problem that the cone which the curve lies on is actually a cone of this type; it could be oriented differently. Also, some people seemed to think that the plane of part (b) and the cone of part (c) were the same object.

(d). The possible intersections of a cone and a plane are a circle, an ellipse, a parabola, a hyperbola, or certain degenerate cases: a single point or two straight lines. Our curve is infinite in extent, so it's not a circle, an ellipse, or a single point. It also cannot have two branches, since it's parametrized by a single continuous function, ruling out a hyperbola and two straight lines. Thus it is a parabola.

A lot of people seemed to think that the only possible intersections of a cone with a plane were a circle or an ellipse. Plotting the curve, although a nice way to check, does not quite suffice; the end of an ellipse, a parabola, and the innermost part of a hyperbola can look very similar, which isn't surprising if you think about them as conic sections since they arise from the intersection of a cone with very similar-looking plane. A proof via the characterization of a plane the set of points equidistant from a point (the focus) and a line (the directrix) is possible – at least one student did it – but it's not recommended because the algebra is painful. A lot of people did something like rewriting the equations defining the curve as

$$x + 1 = \frac{y^2}{4} = z - 1$$

and then looked at projections onto various coordinate planes; I accepted this sort of answer although it's not strictly correct, because we don't know that projections of parabolas in space are parabolas on the plane.