Math 114, solutions to Assignment 4

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October 6, 2007

These are the solutions to the fourth homework assignment. The plain text represents solutions to the problems; the *italic text* represents my comments on the solution.

1 Section 14.3, problems 32-33

In #32, graph a is the curvature $\kappa(x)$, and b is the function $f(x)$. The points where graph a is at zero correspond to the inflection points of b, as they should, since curvature is zero at an inflection point. Furthermore, b cannot be the curvature of a, because a has "cusps" where the curvature should be infinite or discontinuous. So a is the curvature of b.

In #33, the inflection points of b do *not* correspond to zeroes of a, so a cannot be the curvature of b. Thus we conclude that b is the curvature of a.

2 Section 14.3, problem 51a

Following the proof of Theorem 10 in this section: since $\vec{T} = \vec{r}'/|\vec{r}'|$ and $|\vec{r}'| = ds/dt$, we have

$$\vec{r}' = |\vec{r}'|\vec{T} = \frac{ds}{dt}\vec{T}$$

and from the product rule, we have

$$\vec{r}'' = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\vec{T}'.$$

where primes denote differentiation with respect to t. We then observe that

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \kappa\vec{N}s'$$

where the first equality is from the chain rule, and the second equality follows from the first Frenet-Serret formula.

Some people wrote something like "because $ds/ds = 1$" to justify the step that I've justified using the chain rule. Although this is a convenient way to remember the chain rule, it's not true that the chain rule follows from this statement. This *is mostly because we haven't really defined the piece of notation "ds" to mean anything by itself, so the fact that you can "cancel" in the equation*

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$$

doesn't really mean anything. Furthermore, the sorts of things one can "prove" this way won't even be true when we get to partial derivatives.

Substituting this back in, we have

$$\vec{r}'' = s''\vec{T} + s'(\kappa\vec{N}s')$$

which was what we wanted.

A few people have asked me about 51b, 51c, or 51d. These are all the same sort of computation, and are good exercises in working with vectors. If you want more details, let me know and I'd be glad to provide them.

There was a "hint" given as well: "why is this easier after you've done section 14.4?" This is essentially because the formula in question is a renotated version of equation 7 on p. 910, which says that

$$\vec{a} = v'\vec{T} + \kappa v^2\vec{N}$$

where $\vec{a}$ is acceleration and v is speed. It wasn't necessary to write this; rather, it was intended to get you thinkin.

3 Train problem

We can imagine that the train is traveling on a circular track. (This isn't really important – what matters is the instantaneous radius of curvature – but it's a good picture.) If the driver of the train did not step on the brakes, the acceleration of the train at that moment would be entirely in the normal direction, which is either to the east or the west. The normal component of the acceleration must be to the west, since the overall acceleration is northwest. The acceleration due to the change in speed is then due north, i. e. in the reverse of the direction of the train.

We have the formula

$$\vec{a} = v'\vec{T} + \kappa v^2\vec{N}$$

and we want to find the speed v. The normal component of the acceleration is (10 MPH/sec) $\cos\pi/4$, or $5\sqrt{2}$ MPH/sec. The radius of curvature is $\kappa^{-1} = 4\text{mi}^{-1}$. Thus we have

$$5\sqrt{2}\text{MPH/sec} = (4\text{mi}^{-1})v^2$$

and solving for v, we get

$$v = \sqrt{\frac{5\sqrt{2}\text{MPH/sec}}{4\text{mi}^{-1}}}.$$

The units are a bit awkward; we note that 1 MPH/sec = 3600 MPH/hour, and so

$$v = \sqrt{\frac{18000\sqrt{2}\text{MPH/hour}}{4\text{mi}^{-1}}}$$

so $v = \sqrt{4500\sqrt{2}}$ miles per hour, or 79.8 miles per hour.

The big difficulty with this problem for most people was the unit conversion. A lot of people computed the number $\sqrt{5\sqrt{2}/4}$*, which is about 1.3, and claimed that the train was traveling at 1.3 miles per hour. This is not a reasonable speed for a train!*