Math 114, solutions to Assignment 6

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These are the solutions to the sixth homework assignment. The plain text represents solutions to the problems; the *italic text* represents my comments on the solution.

1 Section 15.3, problem 66

Level curves are shown for a function f Determine whether the following partial derivatives are positive or negative at the point P: (a) f_x; (b) f_y; (c) f_{xx}; (d) f_{xy}; (e) f_{yy}.

(a) As we head in the x-direction (to the right), f decreases, so $f_x < 0$.

(b) As we head in the y-direction (up), f increases, so $f_y > 0$.

(c) As we head in the x-direction, x decreases more slowly (the contour lines are further apart), so $f_{xx} > 0$.

(d) As x increases, the contours in the y-direction spread out, indicating that y is increasing more slowly, so $f_{xy} < 0$.

(e) As we head up, the increase of f becomes faster (since the contours are getting closer together), so $f_{yy} > 0$.

I wasn't looking for very verbose explanations here, but I wanted to see something beyond just the answers "positive" and "negative". Most of these were straightforward, with the exception of (d); some people interpreted f_{xy} as the directional derivative in the direction of the line $y = x$.

2 Section 15.5, problem 37 and error estimation

Although most solutions included the solution to problem 37 and then a separate analysis of the error, the calculation and the error analysis are naturally intertwined with each other and I'll present them that way here.

We want to find $\frac{dC}{dt}$. By the chain rule, we have

$$\frac{dC}{dt} = \frac{\partial C}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial C}{\partial D}\frac{\partial D}{\partial T}.$$

Make sure that you write carefully enough that capital T and lowercase t can be distinguished! Although I don't penalize for bad handwriting, if your handwriting *is bad enough that you forget which quantities you're working with, that's a problem.*

Differentiation gives $\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2, \partial C/\partial D = 0.016$. *Although this wasn't explicitly asked, you should think about what the units of these various constants are.* At the time in question, we have $T = 12.8 \pm 0.2$, $D = 7 \pm 0.5$. *I'm assuming I can read the graphs to within a tenth of a square; this may be overly optimistic, but what mattered here was that you handled the error correctly.* Thus, we have $\partial C/\partial T = 3.335$. The *error* in this quantity is given by

$$0.2\left|\frac{\partial}{\partial T}\frac{\partial C}{\partial T}\right|$$

which is just the uncertainty in T, multiplied by the effect that a change in T has on a change in $\partial C/\partial T$. We have $\partial^2 C/\partial T^2 = -0.11 + .00174T$; letting $T = 12.7$ we see that the error in our $\partial C/\partial T$ is $|(0.2)(-0.11 + (0.00174)(12.7)| = 0.018$. So we write $\partial C/\partial T = 3.335 \pm 0.018$.

Next, we must determine $\partial T/\partial t$ at $t = 20$. This is estimated from the graph; we take $(T(25) - T(15))/10$, which is $((12.0 \pm .2) - (13.0 \pm .2))/10$. In the case that the errors are *independent* (as the errors in the quantities "12.0" and "13.0" are), the error in their sum or difference is the square root of the sum of the squared errors, or $\sqrt{(.2)^2 + (.2)^2} = .28$. Thus $T_t(20) = -0.100 \pm 0.028$.

We now compute $\partial C/\partial D$ at $t = 20$; this is $D = 0.016$, exactly, since we are assuming the given formula is error-free.

Finally, $\partial D/\partial t$ at $t = 20$ can be estimated in the same manner as $\partial T/\partial t$, as $(D(25) - D(15))/10$, or $((10 \pm .5) - (5 \pm .5))/10 = 0.50 \pm 0.07$.

So, in the end, we have

$$\frac{dC}{dt} = (3.335 \pm 0.018)(-0.100 \pm 0.028) + (0.016)(0.50 \pm 0.07).$$

The errors in the two factors of the first term are independent. The fractional uncertainty in a *product* is the sum of the uncertainties in the two terms; thus the fractional uncertainty here is $(0.018/3.335) + (0.028/0.100) = 29\%$, and the first term is equal to $(-.34 \pm .10)$. The second term is, more simply, $(.01 \pm .001)$; adding these together, the final answer is $(-.33 \pm .10)$. The units here are (meters per second) per minute.

Don't worry if you didn't get the details of the error analysis; we were kind of vague about what we wanted here. It's more important that you know that you can use calculus to analyze errors than that you can actually reproduce this; different disciplines and different instructors seem to have different tastes in how errors should be analyzed. The main idea is that in a result which depends on several inputs, changing one of the inputs changes the result, and partial derivatives tell us the magnitude of that change.

3 Section 15.6, problem 34

(a) The rate of ascent or descent if we head due south is the directional derivative in the southward direction, which is $-\partial z/\partial y = 0.04y$. At $(50, 80, 847)$ this is 3.2, so the rate of ascent is 3.2; the "units" of this quantity are "meters per meter", or more precisely "vertical meters per horizontal meter".

(b) The rate of ascent of descent heading northwest is the directional derivative in the northwest direction; this direction is given by $\vec{u} = \langle -1/sqrt2, 1/\sqrt{2}\rangle$. *It is essential to normalize this vector!* This is the dot product of ∇f and $\vec{u}$, or

$$\langle -0.02x, -0.04y\rangle \cdot \langle -1/sqrt2, 1/sqrt2\rangle$$

and at $x = 50, y = 80$ this is $-1.1\sqrt{2}$. We are thus *descending* at a rate of $1.1\sqrt{2} = 1.56$ vertical meters per horizontal meters.

(c) The slope is largest in the direction of the gradient, $\langle -1, -3.2\rangle$. The rate of ascent in that direction is the magnitude of the gradient, $\sqrt{1^2 + 3.2^2}$, or approximately 3.35. The angle above the horizontal of the path is $tan^{-1}\sqrt{1^2 + 3.2^2}$, approximately 73.4 degrees. *Note that the similar-looking 72 degrees, which is* $tan^{-1}3.2$*, isn't right; this is the angle of the path with the eastward vector. "East" and "horizontal" aren't the same thing, even though sometimes our drawings might imply that.*