

Math 114, solutions to Assignment 7

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These are the solutions to the seventh homework assignment.

1 Section 15.7, problem 52

We want to show that if $p + q + r = 1$ and p, q, r are nonnegative, that $2pq + 2pr + 2rq \leq 2/3$.

Since $p + q + r = 1$, we can write $r = 1 - p - q$, and then the function we wish to examine is

$$f(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q) = 2(p + q - pq - p^2 - q^2)$$

We want to maximize this function over the region where $0 \leq p, 0 \leq q, p + q \leq 1$. We first check its values along the boundary. If $p = 0$, then we want to maximize $f(0, q) = 2(q - q^2)$, which is maximized at $1/2$; similarly if $q = 0$, we want to maximize $f(p, 0) = 2(p - p^2)$. If $q = 1 - p$, then $f(p, 1 - p) = 2(p - p^2)$ is again maximized at $1/2$. Now, we look for critical points of f within the triangle. We have $f_p = 2(1 - q - 2p)$, $f_q = 2(1 - p - 2q)$; to have both of these equal to zero we need $p = 1/3, q = 1/3$, as can be seen by solving the linear system. This is a critical point in the interior of our region; we compute $f(1/3, 1/3) = 2/3$. Finally, we compute the discriminant $f_{pp}f_{qq} - f_{pq}^2 = (-4)(-4) - (-2)^2 = 12$; the discriminant is positive and f_{pp} is negative, so $2/3$ is a local maximum. The global maximum of f occurs either at a critical point or on the boundary; we see $f = 1/2$ on the boundary, so $f(1/3, 1/3) = 2/3$ is a global maximum.

A lot of people neglected to check the boundaries! This is important. Also, it's possible to solve this problem using Lagrange multipliers, which a few people did.

2 Rocket Science problem, p. 1008

1. In the first stage, we can consider the rocket mass to be M_1 and the payload $M_2 + M_3 + A$; thus we have

$$\Delta V_1 = -c \log \left(1 - \frac{(1 - S)M_1}{M_1 + M_2 + M_3 + A} \right)$$

which simplifies to

$$\Delta V_1 = c \log \left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} \right)$$

giving the first term of the desired formula. The other two terms arise similarly:

$$\Delta V_2 = c \log \left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A} \right)$$

from the second stage, with rocket mass M_2 and payload mass $M_3 + A$, and the third term

$$\Delta V_3 = c \log \left(\frac{M_3 + A}{SM_3 + A} \right)$$

from the third stage, with rocket mass M_3 and payload A .

2. This is basic algebraic manipulation. It's easiest to plug in the values of N_i into the right-hand sides of the equation and see that they equal the left-hand sides. In the case of the first equation, we have

$$\frac{(1-S)N_1}{1-SN_1} = \frac{(1-S) \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A}}{1-S \left(\frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A} \right)}$$

and multiplying through by $SM_1 + M_2 + M_3 + A$ gives

$$\frac{(1-S)(M_1 + M_2 + M_3 + A)}{SM_1 + M_2 + M_3 + A - SM_1 - SM_2 - SM_3 - SA}$$

The denominator factors, giving

$$\frac{(1-S)(M_1 + M_2 + M_3 + A)}{(1-S)(M_2 + M_3 + A)}$$

and the factors of $1-S$ cancel to give the desired result,

$$\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = \frac{(1-S)N_1}{1-SN_1}.$$

Similarly, we get the expressions

$$\frac{M_2 + M_3 + A}{M_3 + A} = \frac{(1-S)N_2}{1-SN_2}, \quad \frac{M_3 + A}{A} = \frac{(1-S)N_3}{1-SN_3}.$$

Now, to get the claimed expression for $\frac{M+A}{A}$ we just multiply these last three expressions together; the product on the left-hand side telescopes, giving

$$\frac{M+A}{A} = \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}$$

where we have used the fact that $M = M_1 + M_2 + M_3$.

Some people started by writing something like

$$\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = \frac{(1 - S)N_1}{1 - SN_1}$$

and then manipulated the right-hand side until it looked like the left-hand side again. This isn't correct, because you're assuming what you're trying to prove. It's a good method for discovering a proof, but it is not the right way to present a proof because it includes the implicit assumption that all your manipulations are reversible. This isn't always true; sometimes "reversing" a chain of manipulations involves things like dividing by zero.

3. We note that $\log(M + A)/A$ is minimized at the same location as M , because we can rewrite it as $\log(1 + M/A)$; the derivative of this function with respect to M is positive, so it increases when M increases and decreases when M decreases. So we can minimize

$$\log \frac{M + A}{A} = 3 \log(1 - S) + \log N_1 + \log N_2 + \log N_3 - \log(1 - SN_1) - \log(1 - SN_2) - \log(1 - SN_3)$$

which we will call $f(N_1, N_2, N_3)$, subject to the constraint that $v_f = c(\log N_1 + \log N_2 + \log N_3)$; we rewrite this constraint as $g(N_1, N_2, N_3) = \log N_1 + \log N_2 + \log N_3 = k$ where $K = v_f/c$.

Setting up the system for Lagrange multipliers, we get $f_1 = \lambda g_1, f_2 = \lambda g_2, f_3 = \lambda g_3, g = k$ where the subscript i denotes differentiation with respect to N_i . This gives the system

$$\begin{aligned} \frac{1}{N_1} \frac{-S}{1 - SN_1} &= \frac{\lambda}{N_1} \\ \frac{1}{N_2} \frac{-S}{1 - SN_2} &= \frac{\lambda}{N_2} \\ \frac{1}{N_3} \frac{-S}{1 - SN_3} &= \frac{\lambda}{N_3} \\ \log N_1 + \log N_2 + \log N_3 &= k \end{aligned}$$

In the first equation we can divide each side by $1/N_1$ (note that N_1 isn't zero, since we can take its logarithm) to get $-S/(1 - SN_1) = \lambda$. We can do the same in the second and third equations, so $-S/(1 - SN_2) = \lambda, -S/(1 - SN_3) = \lambda$. Solving the first of these for N_1 , we get $N_1 = (s + \lambda)/(\lambda s)$; N_2 and N_3 are equal to this same quantity. So $N_1 = N_2 = N_3$; from the third equation we see that $N_1 = N_2 = N_3 = \exp(k/3) = \exp(v_f/3c)$. This is the only solution to the Lagrange multiplier system; it is a minimum of f on physical grounds. The value of f here is seen to be

$$f_{min} = f(\exp(v_f/3c), \exp(v_f/3c), \exp(v_f/3c)) = 3 \log(1 - S) + \frac{v_f}{c} - 3 \log(1 - S e^{v_f/3c})$$

Showing that $\log \frac{M+A}{A}$ and M are minimized at the same place is crucial; otherwise, there's no point in minimizing $\log \frac{M+A}{A}$, since M is what we really care about.

4. We have $f_{min} = \log(1 + M/A)$; thus $M = (\exp(f_{min}) - 1)A$. That is,

$$M = \frac{(1 - S)^3 e^{v_f/c}}{(1 - S e^{v_f/3c})^3} A.$$

5a. The minimum total mass M of the rocket engines is obtained by plugging in $S = 0.2$, $v_f = 17500$ mi/h, $c = 6000$ mi/h into the above equation; we get $M = 89.4A$.

5b. Let

$$b = \frac{(1 - S)e^{v_f/3c}}{1 - S e^{v_f/3c}}.$$

Then we have the equations

$$\frac{M_3 + A}{A} = b, \frac{M_2 + M_3 + A}{M_3 + A} = b, \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = b$$

and so

$$M_3 = (b - 1)A, M_2 = b(b - 1)A, M_1 = b^2(b - 1)A.$$

In the case of the numbers given here, $b = 4.488$; we get $M_3 = 3.5A$, $M_2 = 15.7A$, $M_3 = 70.3A$.

6. This is the same as part 5(b) as written here, except now we have $b = 14.94$, and $A = 500$ lb; thus $M_3 = 6970$, $M_2 = 104000$, $M_1 = 1.56 \times 10^6$, with all units in pounds.

Notice that to get a speed about 40 percent more than in the previous case we need more than thirty times as much fuel! For a three-stage rocket the fuel-to-payload ratio is $b^3 - 1$, where b is defined as above. For an n -stage rocket, the fuel-to-payload ratio is

$$\frac{(1 - S)^n e^{v_f/c}}{(1 - S e^{v_f/nc})^n} - 1$$

which can be seen to approach $\exp\left(\frac{v_f}{c(1-S)}\right) - 1$ as $n \rightarrow \infty$. This is the payload-to-fuel ratio in a hypothetical “infinite-stage” rocket, which essentially jettisons each bit of rocket immediately after it’s been used. In our first case, this is $\exp(17500/(6000)(.8)) = 38.3$, as opposed to the 89.4 we actually got; in our second case, it’s $\exp(24700/(6000)(.8)) = 172$, as opposed to the $b^3 - 1 = 3300$ we actually got, or the 668-to-1 payload ratio we could have achieved with a four-stage rocket. I don’t know why they don’t use four-stage rockets; the reasons for preferring three stages are well outside my expertise.

3 A limit from the rocket problem

You weren’t asked to do this, but look at the quantity which arises in part 4,

$$M = \frac{(1 - S)^3 e^{v_f/c}}{(1 - S e^{v_f/3c})^3} A.$$

As you can imagine, if we had a rocket with more stages, we'd have

$$M = \frac{(1 - S)^n e^{v_f/c}}{(1 - S e^{v_f/nc})^n} A$$

and we might wonder – if there are more stages, what happens to this function? If we plug in larger and larger values of n , M seems to approach some limit. Let $k = e^{v_f/c}$; then we're interested in the limit

$$L = \lim_{n \rightarrow \infty} \frac{(1 - S)^n k}{(1 - S k^{1/n})^n}.$$

Taking logarithms,

$$\log L = \lim_{n \rightarrow \infty} n \log(1 - S) + \log k - n \log(1 - S k^{1/n}).$$

Now, the first and third terms in this sum will both go to infinity as $n \rightarrow \infty$; we might try to use L'Hopital's rule to find this limit. But this will be unpleasant, because we're taking the limit as $n \rightarrow \infty$ and n appears in an exponent. It turns out that it's easier to make the substitution $k^{1/n} = t$, and so $n = (\log k)/(\log t)$. As $n \rightarrow \infty$, $t \rightarrow 1$ – that is, $\lim_{n \rightarrow \infty} k^{1/n} = 1$. So we get

$$\log L = \log k + \lim_{t \rightarrow 1} \frac{\log k}{\log t} (\log(1 - S) - \log(1 - St))$$

which I'll rewrite as $\log L = (1 + P) \log k$, where

$$P = \lim_{t \rightarrow 1} \frac{\log(1 - S) - \log(1 - St)}{\log t}.$$

As $t \rightarrow 1$, the numerator approaches 0 (since $\log 1 - St$ approaches $\log 1 - S$) and the denominator approaches 0 as well (since $\log 1 = 0$.) Thus we can apply L'Hopital's rule to get

$$P = \lim_{t \rightarrow 1} \frac{\frac{-1}{1-St}(-S)}{1/t} = \frac{S}{1-S}$$

and so

$$\log L = \left(1 + \frac{S}{1-S}\right) \log k = \frac{1}{1-S} \log k.$$

Thus we have

$$L = k^{1/S} = \exp\left(\frac{v_f}{c(1-S)}\right)$$

which is the limit we wanted.