Math 114, solutions to Assignment 8

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November 1, 2007

These are the solutions to the eighth homework assignment.

1 Section 16.3 #50.

We can decompose the region D into disjoint regions R_1 and R_2 given by

$R_1 = \{(x, y) : -1 \leq x \leq 1, 1 \leq y \leq 1+x^2\}, R_2 = \{(x, y) : -1 \leq x \leq y^2, -1 \leq y \leq 1\}$.

(Note that R_1 just barely "hangs together", since $1 + x^2 = 1$ when $x = 0$. We then have

$$\iint_{R_1} xydA = \int_{-1}^{1} \int_{1}^{1+x^2} xydydx = \int_{-1}^{1} \left. \frac{xy^2}{2} \right|_{y=1}^{y=1+x^2} dx = \int_{-1}^{1} \frac{x((1+x^2)^2 - 1)}{2} dx$$

and this is the integral of an odd function between symmetric limits; thus it is zero. Similarly,

$$\iint_{R_2} xy\, dA = \int_{-1}^{1} \int_{1}^{y^2} xy\, dx\, dy = \int_{-1}^{1} \left. \frac{x^2 y}{2} \right|_{x=-1}^{x=y^2} dy = \int_{-1}^{\frac{y^5-y}{2}} dy$$

whic is zero, again, since it is the integral of an odd function between symmetric limits. Thus the original integral is zero.

This was straightforward for most people; I don't think anybody came up with this particular decomposition, though. Most people gave an answer which involved three regions, all of what Stewart calls type I; none of the integrals over those regions are zero, but they do add up to zero. This solution is more "elegant", somehow; it only involves two iterated integrals (instead of three) and the symmetry in the various subproblems is more visible.

2 Estimation of integrals

The problem was to "use double integrals to estimate the sum of $1/(x + y)$ over all pairs (x, y) of integers between 1 and 100."

We begin by noticing that the sum

$$\sum_{x=1}^{100}\sum_{y=1}^{100}\frac{1}{x+y}$$

is the Riemann sum for some integral. The most “obvious" integral to try is

$$\int_1^{100}\int_1^{100}\frac{1}{x+y}\,dx\,dy$$

which is a routine calculation:

$$\begin{aligned}\int_1^{100}\int_1^{100}\frac{1}{x+y}\,dx\,dy &= \int_1^{100}\ln(x+y)|_{y=1}^{y=100}\,dx\\ &= \int_1^{100}\ln(x+100)-\ln(x+1)\,dx\\ &= (x+100)\ln(x+100)-(x+1)\ln(x+1)|_{x=1}^{100}\\ &= 200\ln 200-202\ln 101+2\ln 2\\ &= 128.795\ldots\end{aligned}$$

However, as Dr. Pemantle wrote in an e-mail to me, “this integral is over a 99x99 area, so we suspect it isn't one for which the double sum appears as an approximation. However, if we replace 100 by 101, we get the integral

$$\int_1^{101}\int_1^{101}1/(x+y)dxdy$$

If we chop [1,101] x [1,101] into 10,000 squares of side 1 and use the value $f(x_j,y_j)$ in the lower-left corner of each square, then each Delta (area) is exactly 1, so the approximating sum is exactly the sum we were asked to do. The new answer is $-204\ln 3-204\ln 17+202\ln 101=130.1619\ldots$ [...] Note that we will probably do better if we find an integral for which the double sum is a midpoint approximation. Using unit squares centered at integers whose opposite corners are (i-1/2,j-1/2) and (i+1/2,j+1/2), we get

$$\int_{1/2}^{100.5}\int_{1/2}^{100.5}1/(x+y)dxdy.$$

This yields 133.71 which is very close to the 133.64.”

It turns out that it is possible to determine how good an approximation an integral is to a “corresponding” Riemann sum using something called the “Euler-Maclaurin formulas”. These formulas tell us that

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\frac{1}{x+y}=(2\ln 2)n-\ln n-\frac{1}{2}-\gamma+\ln 2+O(n^{-1})$$

where $\gamma = .57721\cdots$ is the so-called Euler-Mascheroni constant, and $O(n^{-1})$ represents some function of n which goes to zero at least as fast as $1/n$ does; in the case where $n = 100$ this tells us that

$$\sum_{x=1}^{100}\sum_{y=1}^{100}\frac{1}{x+y} \approx 200\ln 2 - \ln 100 - \frac{1}{2} - \gamma + \ln 2 = 133.64$$

where the true value is (to two decimal places) 133.63. I may write up something explaining this in more detail.