Math 114, solutions to Assignment 9

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These are the solutions to the ninth homework assignment.

1 Section 16.4 #37.

Part (a) asks you to find the integral $\int_0^\infty x^2 e^{-x^2}\,dx$. We can integrate this by parts, setting $u = x, dv = xe^{-x^2}\,dx$; thus $du = dx, v = -\frac{1}{2}e^{-x^2}$. So we have

$$I = \lim_{t\to\infty} \frac{-t}{2}e^{-t^2}\Big|_0^\infty + \int_0^\infty \frac{1}{2}e^{-x^2}\,dx.$$

This can be rewritten as

$$\left(\lim_{t\to\infty} \frac{-t}{2e^{t^2}}\right) - 0 + \int_0^\infty \frac{1}{2}e^{-x^2}\,dx$$

and the limit is 0 by l'Hopital's rule. Thus we just have

$$I = \int_0^\infty \frac{1}{2}e^{-x^2}$$

and we notice that $\frac{1}{2}e^{-x^2}$ is an even function, so

$$\int_0^\infty \frac{1}{2}e^{-x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{2}e^{-x^2}\,dx. = \frac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx.$$

The last integral in the previous equation is $\sqrt{\pi}$, so we have $I = \frac{1}{4}\sqrt{\pi}$.

In part (b), we are asked to evaluate $\int_0^\infty \sqrt{x}e^{-x}\,dx$. Making the substitution $u = \sqrt{x}$, this becomes

$$\int_0^\infty 2u^2 e^{-u^2}\,du$$

and from the previous problem, $\int_0^\infty u^2 e^{-u^2}\,du = \frac{1}{4}\sqrt{\pi}$; therefore

$$\int_0^\infty \sqrt{x}e^{-x}\,dx = \frac{1}{2}\sqrt{\pi}.$$

2 Section 16.7 # 19.

We can write this region as

$$\{(x,y,z) : x^2 + y^2 \leq 9, 1 \leq z \leq 5 - y\}$$

or, in order to make the bounds of integration more explicit,

$$\{(x,y,z) : -3 \leq x \leq 3, -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}, 1 \leq z \leq 5 - y\}.$$

Thus the integral we want is

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} dz\, dy\, dx.$$

Doing the z-integral gives

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4 - y\, dy\, dx.$$

The y-integral is then

$$\int_{-3}^{3} 4y - \frac{y^2}{2} \Big|_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$$

and evaluating at these values of y gives

$$\int_{-3}^{3} 8\sqrt{9-x^2}\, dx.$$

We can look up this integral in the table of integrals in the text, or use trigonometric substitution; however, it's quickest to notice that

$$\int_{-3}^{3} \sqrt{9-x^2}\, dx$$

is the area of a semicircle of radius 3, namely $9\pi/2$. Thus, our desired integral is $8(9\pi/2)$, or 36π.

Alternatively, we can do this in cylindrical coordinates, writing the region as

$$\{(r,\theta,z) : r \leq 3, 0 \leq \theta \leq 2\pi, 1 \leq x \leq 5 - r\sin\theta\}.$$

This gives the iterated integral

$$\int_{0}^{3} \int_{0}^{2\pi} \int_{1}^{5-r\sin\theta} r\, dz\, d\theta\, dr.$$

Doing the z-integration gives

$$\int_{0}^{3} \int_{0}^{2\pi} (4 - r\sin\theta) r\, d\theta\, dr$$

which can be split up into two integrals

$$\int_0^3 \int_0^{2\pi} 4r \, d\theta \, dr - \int_0^3 \int_0^{2\pi} r^2 \sin\theta \, d\theta dr.$$

Each of these integrals can be factored, giving

$$\int_0^3 4r \, dr \int_0^{2\pi} d\theta - \int_0^3 r^2 \, dr \int_0^{2\pi} \sin\theta \, d\theta.$$

But $\int_0^{2\pi} \sin\theta \, d\theta = 0$, and so wer'e left with

$$\int_0^3 4r \, dr \int_0^{2\pi} d\theta = 2r^2\Big|_0^3 \cdot 2\pi = 2 \cdot 3^2 \cdot 2\pi = 36\pi$$

which is the same answer we got above.

3 Section 16.7 #39.

These are straightforward applications of formulas from the text. The mass is given by

$$m = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2+y^2} \, dz \, dy \, dx.$$

The coordinates of the center of mass are given by

$$\bar{x} = \frac{1}{m} \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} x\sqrt{x^2+y^2} \, dz \, dy \, dx \quad (1)$$

$$\bar{y} = \frac{1}{m} \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} y\sqrt{x^2+y^2} \, dz \, dy \, dx \quad (2)$$

$$\bar{z} = \frac{1}{m} \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} z\sqrt{x^2+y^2} \, dz \, dy \, dx \quad (3)$$

where m is the mass from above. The moment of inertia around the z-axis is

$$M_0 = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} (x^2+y^2)^{3/2} \, dz \, dy \, dx$$

Although you weren't asked to do this, the integrals actually aren't that hard. The mass m is

$$m = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2+y^2} \, dz \, dy \, dx.$$

and doing the z-integral gives

$$\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2+y^2}(4-y) \, dy \, dx.$$

This can be rewritten in polar coordinates to give

$$\int_0^{2\pi} \int_0^3 (4 - r\sin\theta) r^2 \, dr \, d\theta$$

which gives $m = 72\pi$. The integrals for the coordinates of the center of mass and the moment of inertia can be obtained by similar tricks; you should get

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{-27}{40}, \frac{267}{80}\right)$$

and $M_0 = 486\pi$ after a tedious but not conceptually difficult calculation.