Math 170: answers to practice problems

February 28, 2008

These are *answers* to the practice problems posted for the midterm. How to solve these problems was discussed in some detail in lecture and/or recitation; these are just to provide you with some information against which to check yourself.

Please e-mail me at `isabel@math.upenn.edu` to let me know if you find anything you believe to be in error.

1. These are a hyperbola, ellipse, and hyperbola respectively. (Calculate the discrimnant $\Delta = B^2 - 4AC$.)

2. $x^2 + y^2/4$ is a vertical ellipse with foci $(0, \pm\sqrt{3})$ and directrices $y = \pm 4/\sqrt{3}$.

$y = 3x^2 + 2x + 2$ is a parabola with focus $(-1/3, 21/12)$ and directrix $y = 19/12$.

$(x/3)^2 - (y/2)^2 = 1$ is a "horizontal" hyperbola with directrix with foci $(\pm\sqrt{13}, 0)$ and directrices $x = 9/\sqrt{13}$.

For all of these, see the table in the course notes on conic sections.

3. The factorization is $(2x - 3y - 7)(2x - 3y + 3)$; the graph is a pair of lines, with equations $2x - 3y - 7 = 0$ and $2x - 3y + 3 = 0$. (Use the quadratic formula, as suggested.)

4. $\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$; $\sqrt{(0-1)^2 + (1-2)^2 + (0-3)^2} = \sqrt{11}$.

5. The dot product of the vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is $u_1v_1 + u_2v_2 + u_3v_3$; also $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

6. The cosine of each of these angles is $6/\sqrt{40}$; these are equal since the second pair of vectors is just the first pair redrawn in the yz-plane, with the second vector doubled in length.

7. The roots of $x^2 + 9x - 10$ are $1, -10$, and it factors as $(x-1)(x+10)$. The roots of $2x^2 - 3x - 2$ are $2, -1/2$; it factors as $2(x-2)(x+1/2)$, or $(x-2)(2x+1)$. (Use the quadratic formula.)

8. The variable changes are $t = x, t = x + 1/10, t = x + 1/5, t = x + b/2a$. (See the formula given above.)

9. Both of these reduce to $t^3 - 3t + 1$; let $x = t + 4$ and $x = t + 2$ respectively.

10. We get $y = -1/2 \pm \sqrt{-3}/2$. This leads to

$$s = \sqrt[3]{-1/2 \pm \sqrt{-3}/2}$$

(it doesn't matter which one we pick), so

$$t = \sqrt[3]{-1/2 \pm \sqrt{-3}/2} + \frac{1}{\sqrt[3]{-1/2 \pm \sqrt{-3}/2}}$$

and $x = t + 2$ or $t + 4$ depending on which of the cubics from problem 9 we're doing.

11. $x^3 + 6x - 7$ has root $x = 1$; $x^3 - 9x + 28$ has root $x = -4$; $x^3 - 12x + 65$ has root $x = -5$.

12. The complex numbers $a + ib$ satisfying $(a + ib)^3 = 1$ are

$$1, \frac{-1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

13. The square roots of $-i$ are

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

14. We get $f'(x) = 9x^2 - 4$. $f'(0) = -4$ and the tangent line at $(0, -1)$ is $y + 1 = -4(x - 0)$; $f'(2/3) = 0$ so the tangent line at $(2, -7)$ is $y = -7/9$.

15. We get $f'(x) = 4x$. The tangent lines at $(2, 7)$ and $(0, -1)$ are $y = 8x - 9$ and $y = -1$, respectively.

16. The area under $y = -x^2 + 4$ between $x = 0$ and $x = 1$ is $11/3$.

17. $f'(x) = 6x + 1$; the area under $f'(x)$ between $x = 0$ and $x = t$ is $3t^2 + t$, which is just the original function $f(x)$ with a different variable name. This is because differentiation (finding slopes) and integration (taking areas) are inverses of each other.

18. The following are copied from the course notes: Fundamental theorem of arithmetic: Every positive integer can be written in a unique manner as a product of primes. Fundamental theorem of algebra. Every polynomial with complex coefficients has a complex root. Fundamental theorem of the calculus: The two operations of taking the derivative function that computes the slope of the tangent line at the point x and computing the area under a curve are inverse to each other

The fundamental theorem of arithmetic was the one we proved completely. The fundamental theorem of algebra we just stated without proof. The fundamental theorem of calculus we only proved when the function in question is a polynomial.