Mathematics 170: Ideas in Mathematics

Advice on Homework 1

I want to put here, in writing, problems that are similar to problems 3 and 5, so that you get some idea of how to solve them. These are also meant to be examples of the sort of writing I'm looking for; I'm not looking for great masterpieces, but as you can see I use the English language to explain my thought process.

Problem 3 was: Consider fifteen points arranged in an equilateral triangle. (You should have in mind the rack that holds the balls at the beginning of a game of pool.) Prove that if you color each of these points either black or white, there will be an equilateral triangle with its three corners all the same color. You could in theory just show all the colorings, but there are too many! I'm looking for a well-organized argument.

A similar problem is the following: Consider nine points in a straight line, numbered 1 through 9. Prove that if you color each of these points either black or white, there are three "evenly spaced" points which are the same color. By "evenly spaced" I mean the points $n, n+k, n+2k$ for some positive integers n and k.

Here's how you could solve this. We'll try to color the points in such a way that we avoid making an evenly spaced triple of the same color for as long as we can, and we'll see that eventually this fails.

First, if there's a way to color the points so that there aren't three evenly spaced points with the same color, then there's a way to do this such that 1 is black. So assume 1 is black.

Now 2 must be either black or white.

If 2 is black, then 3 must be white so that we don't have the black triplet $(1, 2, 3)$. Then 4 can be either white or black. Say 4 is white; then we have the arrangement $BBWWxxxxx$, where B represents black, W represents white, and x can be either color. Now 5 must be black: if 5 were white then we'd have the white triplet $(3, 4, 5)$. Now 8 must be white to avoid the black triplet $(2, 5, 8)$, and 9 must be white to avoid the black triplet $(1, 5, 9)$. So we have $BBWWBxxWW$. Then 7 is black, to avoid the white triplet $(7, 8, 9)$, and we have $BBWWBxBWW$. Finally, if 6 is black here then $(5, 6, 7)$ is a black triplet; if 6 is white then $(3, 6, 9)$ is a white triplet.

Still assuming 2 is black (and 3 is white), let 4 be black. Then we have $BBWBxxxxx$. To avoid the black triplets $(1, 4, 7)$ and $(2, 4, 6)$, 6 and 7 must be white. So we have $BBWBxWWxx$. Now to avoid the white triplets $(5, 6, 7)$ and $(6, 7, 8)$, we must make 5 and 8 both black; but then we get the black triplet $(2, 5, 8)$.

So 2 can't be black. Thus if there *is* an allowable coloring, 2 is white, giving $BWxxxxxxx$. 3 can be either white or black. If 3 is black, then we have $BWBxxxxxx$; thus 5 has to be white to avoid the black triplet $(1, 3, 5)$, giving $BWBxWxxxx$. So 8 is black to avoid the white triplet $(2, 5, 8)$, giving $BWBxWxxBx$. None of the colors are forced at this point, so we'll choose the color of 4 freely. Let 4 be black; then 7 must be white to avoid the white triplet $(1, 4, 7)$, and we have $BWBBxxWBx$. So 6 must be white to avoid the black triplet $(4, 6, 8)$, giving $BWBBxWWBx$. No matter how we color 5 there will be the sort of triplet we don't want. If instead we let 4 be white, then we have the arrangement $BWBWWxxxx$;

then in turn 6 is forced to be black, and 9 is forced to be white, giving $BWBWWBxxW$. Now 7 must be black to avoid the white triplet $(5, 7, 9)$, and there's no right way to color 8. Therefore we can't have 4 white either, in this situation.

So any possible coloring has 2 white and 3 white, and we have $BWWxxxxxx$. 4 is forced to be black, and then 7 is forced to be white, giving $BWWBxxWxx$. Then 5 must be black to avoid the white triplet $(3, 5, 7)$, giving $BWWBBxWxx$. Then 6 is white and 8 is black, giving $BWWBBWWBx$. If we color 9 black then $(1, 5, 9)$ is a black triplet; if we color it white then $(3, 6, 9)$ is a white triplet. Thus the coloring with 2 white and 2 white isn't possible, and thus there is no possible coloring.

The reasoning for problem 3 is basically similar to this. Let's say we gave the balls names

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    A
   B C
  D E F
 G H I J
K L M N O
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and colored A white, and K and O black. Now B and C can't both be white. If they're both black, then you can see that E and N must be white; what happens then?

If B is black and C is white, then N is still forced to be white; now what? Nothing is forced, so you can see (for example) what happens in the case when M is black and then in the case when M is white. (I don't guarantee that this is the quickest path to an answer, just that it's the one I found.)

Problem 5 was: Prove by induction that $1^2 + 2^2 + ... + n^2 = n(n + 1)(2n + 1)/6$.

I'll do a similar problem: Prove by induction that $1 + 7 + 19 + \ldots + (3n^2 - 3n + 1) = n^3$.

First, do the base case. This is clearly true when $n = 1$; then it's just $1 = 1$.

Next, do the inductive step. Assume that $1 + 7 + \ldots + (3n^2 - 3n + 1) = n^3$; we want to show that

$$1 + 7 + \ldots + (3n^2 - 3n + 1) + (3(n + 1)^2 - 3(n + 1) + 1) = (n + 1)^3$$

which is the result we want to show, with n replaced with $n + 1$.

This is straightforward algebra. By the assumption, $1 + 7 + \ldots + (3n^2 - 3n + 1) = n^3$, so the left-hand side of the equation above is

$$n^3 + (3(n + 1)^2 - 3(n + 1) + 1).$$

Expanding this polynomial gives

$$n^3 + (3(n^2 + 2n + 1) - 3(n + 1) + 1)$$

which equals

$$n^3 + (3n^2 + 6n + 3 - 3n - 3 + 1)$$

and collecting like terms gives

$$n^3 + 3n^2 + 3n + 1$$

which is $(n + 1)^3$.