Mathematics 170: Ideas in Mathematics

Solutions to Homework 4

1(a). Let $R = 5.63121212\ldots$. Then $100R = 563.121212\ldots$. Subtract these; the difference is $99R = 557.49$. So $R = 557.49/99$; multiplying the top and bottom by 100, we get $R = 55749/9900$.

1(b). We have

$$R = 5 + \frac{6}{10} + \frac{3}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} + \cdots$$

and combining pairs of terms, we get

$$R = 5 + \frac{6}{10} + \frac{3}{10^2} + \left(\frac{12}{10^4} + \frac{12}{10^6} + \frac{12}{10^8} + \cdots \right).$$

The part in parentheses is a geometric series, with first term $12/10^4$ and ratio $1/10^2$; thus it sums to $(12/10^4)/(1 - 1/10^2)$. So

$$R = 5 + \frac{6}{10} + \frac{3}{10^2} + \frac{12/10^4}{99/100}$$

which after some arithmetic works out to 55749/9900.

2. $2/13 = 0.153846153846\ldots$. After we get the first digit 6, the remainder is 2. Since 2 was the "remainder" when we began the process, we know that the division process will repeat itself; thus we can stop.

3. Assume that $0.246810121416\ldots$ were a rational number. Then its decimal expansion would be eventually repeating. Call the length of the repeating part L. Now since the numbers $2, 22, 222, 2222, \ldots$ are even, the strings of digits $2, 22, 222, 2222, \ldots$ appear in the decimal expansion of this rational number. In particular, the string of digits consisting of the digit 2 repeated L times must appear. But this takes up the entire repeating part; thus the repeating part must consist entirely of 2s, which contradicts the way that the number is constructed.

4. Given two real numbers, either there's an integer between them, or there isn't. If there is an integer between them, that's the rational number we're looking for. If there is not an integer between them, then we can write them as

$$n.c_1c_2c_3\ldots, n.d_1d_2d_3\ldots$$

where n is some integer, and the cs and ds are decimal digits; assume $n.c_1c_2c_3\ldots < n.d_1d_2d_3\ldots$. Then there's some k such that $c_1 = d_1, c_2 = d_2, \ldots, c_{k-1} = d_{k-1}$, and $c_k < d_k$; that is, the first $k-1$ digits after the decimal point are the same in both real numbers, and the kth digit is smaller in the first one. Then take $n.d_1d_2\ldots d_k$; that is, cut off the second number just after the first digit that disagrees with the first number. This is the number we want.

For example, say we have the two real numbers $3.141592\ldots$ and $3.142857\ldots$; these differ for the first time at the third digit, so take 3.142 to be the rational number between them.

(Strictly speaking, this doesn't work if $n.d_1d_2d_3\ldots$ ends in infinitely many zeroes, because then truncating it doesn't change it. So we should adopt the rule that we use the representation of this number that ends in infinitely many 9s; for example, we write $2.4599999\ldots$ instead of 2.46.)

5. Map the natural number n to the even number $2n$, for each n.

6. The cardinality of the balls in at least one of the barrels must be infinite. If the cardinality of the balls in each barrel was finite, then the cardinality of all the balls put together would also be finite.

7. The set of all triplets of natural numbers is countable.

One way to see this is to remember that a product of two countable sets is countable. Thus, the set of all pairs of natural numbers is countable (as we saw in class), and the product of this set with the natural numbers is countable; the latter set is the triplets. We could construct a list of the triplets by first generating a list of the pairs of natural numbers,

$$(1,1),(2,1),(1,2),(3,1),(2,2),(1,3),(4,1),(3,2),(2,3),(1,4)\ldots$$

and then creating a list which consisted of the first element of this list, followed by 1; then the second element, followed by 1; then the first element, followed by 2; then the third element, followed by 1; and so on. This gives

$$(1,1,1),(1,2,1),(1,1,2),(2,1,1),(1,2,2),(1,1,3),(1,3,1),(2,1,2),(1,2,3),(1,1,4),$$

Another way to see this is to note that there are finitely many triplets of natural numbers summing to each of $3,4,5,6,\ldots$, and the union of countably many finite sets is countable. Thus we could construct a list of the triplets by first writing down all those that sum to 3, then to 4, then to 5, and so on; such a list would begin

$$(1,1,1),(2,1,1),(1,2,1),(1,1,2),(3,1,1),(2,2,1),(2,1,2),(1,3,1),(1,2,2),(1,1,3),(4,1,1),(3,2,1),\ldots$$

These are not the only possible lists.

8. If we had n circles, then we could color them in 2^n ways. For example, if there were four circles to be colored, then we could color them in 16 ways. 2^n is so much larger than n that it starts to seem reasonable that maybe $2^{|\mathbb{N}|}$ is larger than $|\mathbb{N}|$, where $|\mathbb{N}|$ is the cardinality of the natural numbers. In fact it is; the set of all possible circle colorings has the same cardinality as the set of *real* numbers, which is much larger than the cardinality of the naturals. (Circle colorings, in fact, are secretly just binary expansions of real numbers.)