Mathematics 170: Ideas in Mathematics

Homework 5

This assignment is due Wednesday, June 17, 2009, at the beginning of class. Please hand the homework in in class. If you can't make it to class, place it in my mailbox in the departmental office (DRL 4W1) or bring it to my office (DRL 4N27). You are allowed to talk about the homework with each other, but please write it up alone.

1. Suppose you had infinitely many people, each one wearing a uniquely numbered button: $1, 2, 3, 4, 5, \ldots$. Imagine each person has a penny. You give each person a penny; then you ask each person to flip his or her penny at the same time. Then you ask them to shout out, in order, what they flipped (H for heads and T for tails). For example, you hear $HHTHTTTHTTTHTTH\ldots$ or $TTTHTTHHTHHHHTT\ldots$. Does the set of possible outcomes have the same cardinality as the natural numbers? Justify your answer. (B+S 3.3.14)

2. Say that in Cantor's proof of the uncountability of the reals, we construct a real number M not in our countable list of real numbers $r_1, r_2, \ldots$ in the following way: if the kth digit of r_k after the decimal point is 2 then the kth digit of M after the decimal point is 9, and otherwise the kth digit of M is 2. Then it is possible that the real number M appears on our list. Provide a list of real numbers for which this happens. (Recall that $0.1999\ldots = 0.2$.) (B+S 3.3.21; modified a bit because the presentation of Cantor's proof in the text is different from the one done in class)

3. Suppose that Words is the set defined by Words = { all, you, infinity, found, them, search, the, it }. Consider the following pairing of elements of Words with elements of P(Words):

Elements of Words	Elements of P(Words)
all	{ all, infinity, found }
you	{ it, search, them }
infinity	{ all, them, infinity }
found	{ you, the, it }
them	{ found, them }
search	{ all, infinity, search }
the	{ the, search }
it	{ infinity, you, all }

Using the idea of Cantor's proof of the uncountability of the reals, describe a particular element of P(Words) that is not on this list. (B+S 3.4.13)

4. Let S be the set of all natural numbers that are describable in English words using no more than 50 characters (so, 240 is in S since we can describe it as "two hundred forty", which requires fewer than 50 characters). Assuming that we are allowed to use only the 27 standard characters (the 26 letters of the alphabet and the space character), show that there are only finitely many numbers contained in S. (In fact, perhaps you can show that there can be no more than 27^{50} elements in S.) Now, let the set T be all those natural numbers not in S. Show that there are infinitely many elements in T. Next, since T is a collection of natural numbers, show that it must contain a smallest number. Finally, consider the smallest number contained in T. Prove that this number must simultaneously be an element of S and not an element of S – a paradox! (B+S 3.4.20)

5. Suppose we used our failed perfect shuffling of digits to mix the digits of the numbers (x, y) that describe a point on the square to get a (almost) one-to-one correspondence with the points on a line segment.

(a) What point on the square would be paired with the point $0.120001000100010001\ldots$? With which point on the line does that point actually get paired? Is this a problem? Explain.

(b) Repeat (a) for $0.12001001001001\ldots$.

(B+S 3.5.11-12)

6. Prove that the cardinalities of points in the following two geometrical objects are equal (these objects are made up of little line segments – so they have no thickness)

T L

(B+S 3.5.17)