Mathematics 170: Ideas in Mathematics

Homework 6

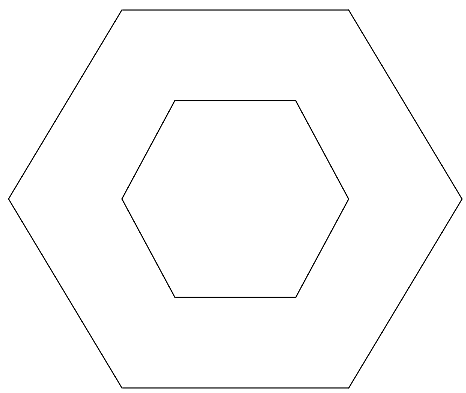
This assignment is due Monday, June 22, 2009, at the beginning of class. Please hand the homework in in class. If you can't make it to class, place it in my mailbox in the departmental office (DRL 4W1) or bring it to my office (DRL 4N27). You are allowed to talk about the homework with each other, but please write it up alone.

1. Consider the closed polygonal curve at the *left* in Section 4.2, Problem 7.

(a) According to the Art Gallery Theorem, how many guards (stationary, in swiveling chairs) will be required to guard the entire gallery?

(b) Triangulate the gallery and show where the guards must be placed.

2. Consider the gallery illustrated below:



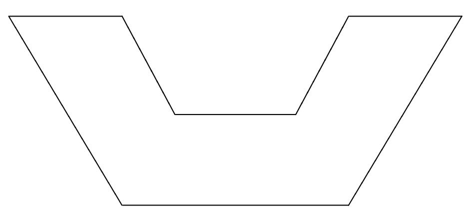
Note that the hexagonal "hole" in the middle is not part of the gallery.

(a) Split up the gallery into two simple closed polygonal regions. (There is more than one way to do this.)

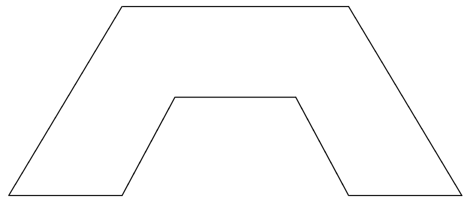
(b) For each of the regions in part (a), use the procedure from the Art Gallery Theorem to position guards so that they guard that region.

(c) Combine your answers from (b) to show how to position guards to guard the entire region.

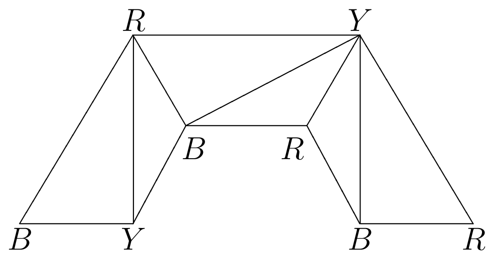
Solution. (a) One way to split the region up is into the two eight-sided regions:



and

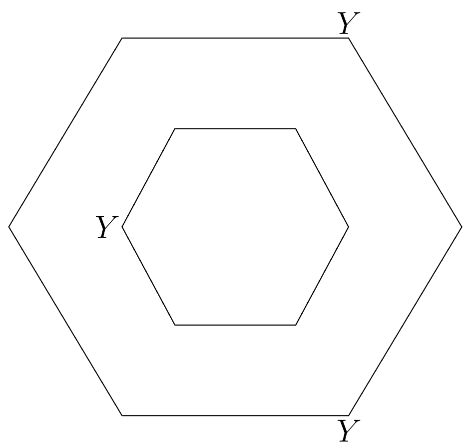


(b) These are each eight-sided regions; they can thus each be guarded with $[8/3] = 2$ guards. In the case of the top region, we have the triangulation and coloring



and we can guard this region by placing guards at the two points labeled Y. Reflecting this over a horizontal line gives a triangulation and coloring of the bottom region.

(c) We can combine these two results to give a means of guarding the original region using four guards. However, two of those guards will be in the same place! In fact, for the original hexagon-with-a-hole region we can position the guards as illustrated below:



3. Draw a picture illustrating the fact that, although we can place 2 regular pentagons (5-sided polygons) and a regular decagon (10-sided polygon) around a single vertex, we cannot extend this to a tessellation of the place which has this arrangement at every vertex.

4. Consider the truncated cube, which is obtained from the ordinary cube by cutting off each corner. (See the picture on p. 370 of the text, or on Wikipedia.)

(a) How many faces does it have? Edges? Vertices? Your answer should refer back to the (ordinary) cube to make the counting easier; recall that the ordinary cube has 6 faces, 12 edges, and 8 vertices.

(b) How many rotational symmetries does it have? Justify your answer.

(c) Your answer to (b) should be a number we've seen before in a similar context. Where have we seen this number before, and why are we seeing it again?

Solution (a) There is one face for each face of the cube and one face for each vertex of the cube, so $6 + 8 = 14$ faces. There is one edge for each edge of the cube, plus three edges for each vertex of the cube, so $12 + (3)(8) = 36$ edges. There are three vertices for each vertex of the cube, so $(3)(8) = 24$ vertices.

(b) There are 24 rotational symmetries. One way to see this is to notice that deciding which vertex a single vertex is taken to by a rotation uniquely specifies the rotation. There are 24 vertices, thus 24 rotational symmetries.

(c) This is the same as the number of symmetries of the cube; in a sense a truncated cube is just a cube where the vertices have been "made larger", i. e. expanded into small triangles.

5. Take a cube. Put a point in the middle of each face. Now draw straight lines to the middles of each of the sides of that fact, producing a plus sign + on each face. The kinked line that goes from the center of one face to the center of an adjacent face forms a bent edge on this cubical world. Thus we have created eight "bent" triangles whose vertices are the centers of the faces of the cube.

(a) What is the sum of the angles for each of those triangles?

(b) What is the sum of all the angles of all the triangles, and how does this compare to the sum of all the angles of eight ordinary triangles in the plane? (B+S 4.6.32)

Solution. (a) Each of these triangles has three right angles, for an angle sum of 270 degrees.

(b) The total angle sum is $8 \cdot 270 = 2160$ degrees, which is 720 degrees more than the total angle sum of eight ordinary plane triangles, $8 \cdot 180 = 1440$ degrees. The difference of 720 degrees is the total angle defect of a polyhedron, as expected.

6. Repeat problem 5 for either the tetrahedron or the dodecahedron. (B+S 4.6.33 or 34)

Solution for the tetrahedron. (a) Each triangle has three angles. Three of these angles meet at the center of each face, so each angle is $360/3 = 120$ degrees. The angle sum of each triangle is 360 degrees. (b) The sum of the angles of all four triangles is $4 \cdot 360 = 1440$ degrees, which is 720 degrees more than the 720-degree total for four plane triangles.

Solution for the dodecahedron. (a) Each triangle has three angles. Five of these angles meet at the center of each face, so each angle is $360/5 = 72$ degrees. The angle sum of each triangle is 216 degrees. (b) The sum of the angles of all twenty triangles is $20 \cdot 216 = 4320$ degrees, which is 720 degrees more than the 3600-degree total for twenty plane triangles.