Mathematics 170: Ideas in Mathematics

Homework 7

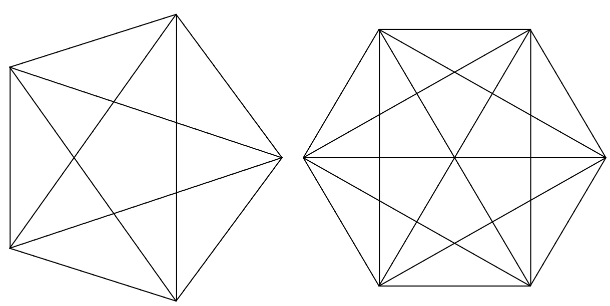
1. (This is a repeat of problem 7 on HW 6.) We saw how to build cubes in all dimensions; how about triangles? A 0-dimensional triangle is just a point. A 1-dimensional triangle is a line segment; you know what a 2-dimensional triangle looks like; a 3-dimensional triangle is a tetrahedron. What is the pattern? We take the triangle we just created and then add a new point in the next dimension "above" the triangle. If we draw new edges from the vertices of the triangles to our new point, then we have a triangle one dimension higher.

(a) Sketch a 4-dimensional "triangle" and a 5-dimensional "triangle".

(b) Fill in a table which shows the number of vertices, edges, 2-dimensional faces, and 3-dimensional faces for the "triangles" in 1, 2, 3, 4, and 5 dimensions.

(c) Guess formulas which give the number of vertices, edges, 2-dimensional faces, and 3-dimensional faces for an n-dimensional "triangle". (B+S 4.7.16)

Solution. (a) Many drawings are possible. But any drawing of a four-dimensional "triangle" should have five vertices all of which are connected to each other by edges; any drawing of a five-dimensional "triangle" should have six vertices all of which are connected to each other by edges. Examples are



(b)

dimension	vertices	edges	2-faces	3-faces
1	2	1	0	0
2	3	3	1	0
3	4	6	4	1
4	5	10	10	5
5	6	15	20	15

(c) The number of vertices of the n-dimensional "triangle" is clearly $n+1$.

Now, when we add a vertex to the $(n-1)$-dimensional "triangle" to create the n-dimensional "triangle", we also add n edges connecting that vertex to each vertex of the $(n-1)$-dimensional triangle. So the n-dimensional triangle has n more edges than the $(n-1)$-dimensional triangle; in particular the number of edges is $1+2+3+\ldots+n$. This is in fact $n(n+1)/2$; call this number $E(n)$

The 2-dimensional faces of the n-dimensional triangle can also be counted. Some of them are 2-D faces of the $(n-1)$-dimensional triangle. The others correspond to edges of the

$(n-1)$-dimensional triangle: each edge gives rise to a face. So the number of 2-dimensional faces of the n-triangle is the number of 2-faces of the $(n-1)$-triangle, plus the number of edges. That is, letting $F_2(n)$ denote the number of 2-faces of the n-triangle,

$$F_2(n) = F_2(n-1) + E(n-1)$$

and working backwards,

$$F_2(n) = 1 + 3 + 6 + \ldots + (n-1)n/2.$$

This is in fact $n(n+1)(n-1)/6$, but this is hard to guess.

Similarly, 3-faces of the n-triangle come from 3-faces of the $(n-1)$-triangle and 2-faces of the $(n-1)$-triangle. Letting $F_3(n)$ denote their number,

$$F_3(n) = F_3(n-1) + F_2(n-1).$$

In fact, all of these numbers are binomial coefficients, the numbers from Pascal's triangle! This occurs since there is a face of an n-triangle corresponding to *any* set of its vertices we might pick.

2. Consider the five regular solids with their vertices cut off. These objects are called *truncated solids*. For each truncated solid, count the number of vertices, edges, and faces, and verify that the Euler characteristic is correct. (See p. 370 of the text for pictures. Also, note that the things you should be "counting" here include those vertices, edges, and faces which are hidden in the drawings.) (B+S 5.3.22)

Solution. For the truncated tetrahedron, there are three vertices where each vertex of the original tetrahedron used to be. Thus there are $4 \cdot 3 = 12$ vertices. The six edges of the original tetrahedron remain, and in addition there are three "new" edges surrounding each vertex of the original tetrahedron, giving $6 + 4 \cdot 3 = 18$ edges. Finally, the number of faces is the number of faces of the original tetrahedron, plus one new face for each vertex, giving $4 + 4 = 8$ faces. The Euler characteristic is therefore $12 - 18 + 8 = 2$.

This pattern holds in general. The number of vertices of a truncated solid is the number of vertices of the original solid, times the number of faces meeting at each vertex of the original solid. The number of edges of a truncated solid is the number of edges of the original solid, plus the number of vertices of the *truncated* solid which we have already computed. Finally, the number of faces of the truncated solid is the number of faces of the original solid, plus the number of vertices of the original solid. This gives the values

solid	vertices	edges	faces
trunc. tetrahedron	$4 \cdot 3 = 12$	$6 + 4 \cdot 3 = 18$	$4 + 4 = 8$
trunc. cube	$8 \cdot 3 = 24$	$12 + 8 \cdot 3 = 36$	$6 + 8 = 14$
trunc. octahedron	$6 \cdot 4 = 24$	$12 + 6 \cdot 4 = 36$	$8 + 6 = 14$
trunc. dodecahedron	$20 \cdot 3 = 60$	$30 + 20 \cdot 3 = 90$	$12 + 20 = 32$
trunc. icosahedron	$12 \cdot 5 = 60$	$30 + 12 \cdot 5 = 90$	$20 + 12 = 32$

We can easily check that $V - E + F = 2$ for each solid. Note that the numbers which are duplicated between the truncated cube and truncated octahedron, and between the truncated dodecahedron and truncated icosahedron, are not duplicated by coincidence; this is yet another consequence of duality.

3. Three hollowed, triangular prisms are used to make a torus. Carefully count the number of vertices, faces, and edges for this prismatic torus. Compute the Euler characteristic for this torus. (B+S 5.3.29)

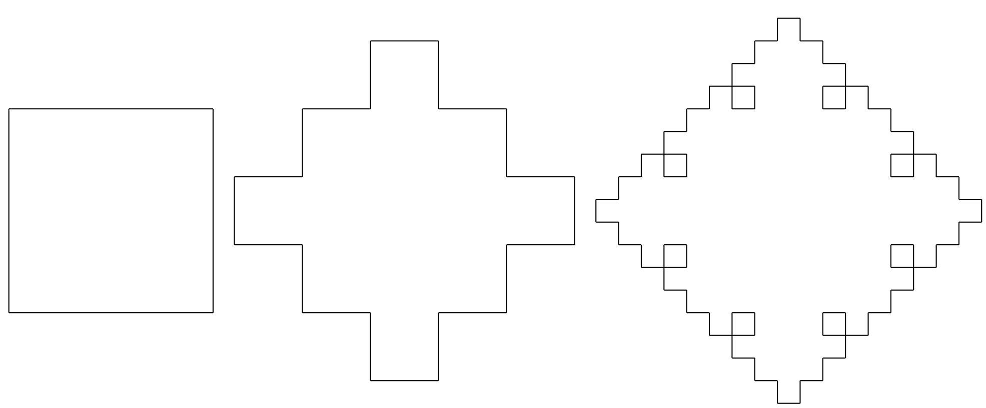
Solution. This torus has nine vertices, six on the "outside" of the hole and three on the "inside". It has nine faces, three arising from each of the three original triangular prisms. It has 18 edges – these can be decomposed into three triangles going around the torus the "long" way and three triangles going around the short way. So the Euler characteristic is $V - E + F = 9 - 18 + 9 = 0$.

4. Carefully count the number of vertices, faces, and edges for a two-holed torus. One way to view this two-holed torus is as two copies of the torus in problem 3, with one side removed from each and then the open edges glued together. This operation is called the *connected sum.* Compute the Euler characteristic for this two-holed torus. (B+S 5.3.30)

Solution. Glue two copies of the torus together along a four-sided face. The number of vertices of the resulting torus is twice the number of vertices of the original torus, minus the number which disappear in the gluing operation. In the gluing operation four vertices disappear – four pairs of corresponding vertices on the two copies become single vertices. So there are $2 \cdot 9 - 4 = 14$ vertices. Similarly, four edges disappear in the gluing operation, so there are $2 \cdot 18 - 4 = 32$ edges. Finally, two faces are removed in the course of the operation, so there are $2 \cdot 9 - 2 = 16$ faces. The Euler characteristic is thus $14 - 32 + 16 = -2$, as it is for any two-holed torus.

5. Recall that in class we constructed the Koch snowflake, a shape which had finite area but infinite perimeter, by starting with an equilateral triangle, replacing each edge with four edges of one-third the length in a certain way, and repeating this process infinitely many times. (See p. 435 of the text for a picture of what happens to *one* edge when we do this.)

We can construct a similar "square Koch snowflake" by beginning with a square and replacing each edge with *five* edges of one-third the length, such that the three middle edges form a square. (Note that the boundary of this snowflake actually touches itself at various points.) The first few iterations of this appear as follows.



Assume that the original square has side length 1.

(a) What is the length of the nth curve in this series? (Note that the nth curve in this series is made up of a large number of short segments of the same length; you should find the length of each short segment and the number of them.)

Solution. The first curve consists of four segments; in each iteration after that there are five times as many curves, so there are a total of $4 \cdot 5^{n-1}$ segments in the nth curve in the series. The segments in each curve are one-third the length of those in the previous one, so the length of each segment in the nth curve is $(1/3)^{n-1}$. Thus the total length is $4 \cdot (5/3)^{n-1}$.

(b) What is the area enclosed by the nth curve in this series? (Note that the nth curve in this series is obtained from the $(n-1)$st curve by adding a large number of small squares. For example, we obtain the second curve from the first one by adding four squares, each of which has side length 1/3 and thus area 1/9. So the area enclosed by the second curve is $1 + 4/9$.) Your answer should involve summing a geometric series.

Solution. To obtain the nth curve from the $(n-1)$st curve, note that the $(n-1)$st curve consists of $4 \cdot 5^{n-2}$ line segments of length $(1/3)^{n-2}$. On each of these line segments we will place a square of side length $(1/3)^{n-1}$, and thus of area $(1/9)^{n-1}$. So to get from the $(n-1)$st curve to the nth curve, we add squares having total area $4 \cdot 5^{n-2} \cdot (1/9)^{n-1}$; we can rearrange this to get $(4/5) \cdot (5/9)^{n-1}$.

The area enclosed by the nth curve in this series (for $n \geq 2$) is therefore

$$1 + (4/5)(5/9) + (4/5)(5/9)^2 + \cdots + (4/5)(5/9)^{n-1}$$

and all the terms but the first make up a geometric series. We can rewrite this as

$$1 + \left[(4/9) + (4/9)(5/9) + (4/9)(5/9)^2 + \cdots + (4/9)(5/9)^{n-2}\right]$$

and recall that $a + ar + \ldots + ar^k = a(1 - r^{k+1})/(1 - r)$. With $a = 4/9, r = 5/9, k = n - 2$, the area enclosed is therefore

$$1 + 4/9 \left(\frac{1 - (5/9)^{n-1}}{1 - 5/9} \right)$$

This can be simplified to give

$$1 + (1 - (5/9)^{n-1})$$

and if n is very large this is very nearly 2. In fact, the area enclosed by this curve is very close to being a square of side length $\sqrt{2}$.

6. Start with the point 0. Flip a coin. If it comes up heads, move 2/3 of the way toward 1, if it comes up tails, move 2/3 of the way to 0 from wherever you are at the time. Repeat forever. The points you find are drawing a picture of the Cantor set. Verify that any point in the Cantor set will move to another point in the Cantor set under the coin-flipping-and-moving process. (B+S 6.3.26)

Solution. Moving two-thirds of the way towards 1 takes the point x to $(2 + x)/3$. To see this, note that the distance from x to 1 is $1 - x$; the distance from $(2 + x)/3$ to 1 is

$$1 - \frac{2 + x}{3} = \frac{1 - x}{3}.$$

Moving two-thirds of the way towards 0 takes the point x to $x/3$.

So we need to show that if x is in the Cantor set, then so are $(2 + x)/3$ and $x/3$. But we did this in class, by looking at the ternary expansion of x.