6.1

4. If the curve is $\mathbf{x}(t) = (\cos 4t,\, \sin 4t,\, 3t)$ then its derivative is $\mathbf{x}'(t) = (-4\sin 4t,\, 4\cos 4t,\, 3)$. The arclength differential is

$$\begin{aligned} ds &= \|\mathbf{x}'(t)\|\, dt \\ &= \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 3^2}\, dt \\ &= \sqrt{16\sin^2 4t + 16\cos^2 4t + 9}\, dt \\ &= \sqrt{16+9}\, dt = \sqrt{25}\, dt = 5\, dt. \end{aligned}$$

Now we can integrate

$$\begin{aligned} \int_{\mathbf{x}} f\, ds &= \int_0^{2\pi} \left(3\,(\cos 4t) + (\cos 4t)\,(\sin 4t) + (3t)^3\right) 5\, dt \\ &= 15\int_0^{2\pi} \cos 4t\, dt + 5\int_0^{2\pi} \sin 4t \cos 4t\, dt + 135\int_0^{2\pi} t^3\, dt. \end{aligned}$$

Use the substitution $u = 4t$ to integrate

$$\int_0^{2\pi} \cos 4t\, dt = \tfrac{1}{4}\sin 4t\Big|_0^{2\pi} = 0$$

and use $u = \sin 4t$ to integrate

$$\int_0^{2\pi} \sin 4t \cos 4t\, dt = \tfrac{1}{8}\sin^2 4t\Big|_0^{2\pi} = 0.$$

Then all that's left is

$$\int_{\mathbf{x}} f\, ds = 135\int_0^{2\pi} t^3\, dt = \tfrac{135}{4} t^4\Big|_0^{2\pi} = 540\pi^4.$$

6. The given path is

$$\mathbf{x}(t) = \begin{cases} (2t,\, 0,\, 0) & \text{if } 0 \le t \le 1, \\ (2,\, 3t-3,\, 0) & \text{if } 1 \le t \le 2, \\ (2,\, 3,\, 2t-4) & \text{if } 2 \le t \le 3. \end{cases}$$

Its derivative is

$$\mathbf{x}'(t) = \begin{cases} (2,\, 0,\, 0) & \text{if } 0 < t < 1, \\ (0,\, 3,\, 0) & \text{if } 1 < t < 2, \\ (0,\, 0,\, 2) & \text{if } 2 < t < 3. \end{cases}$$

Note that $\mathbf{x}'(t)$ is undefined when t is 0, 1, 2, or 3. To integrate along $\mathbf{x}$, split it up into differentiable pieces:

$$\int_{\mathbf{x}} f\,ds = \int_{t=0}^{t=1} f\,ds + \int_{t=1}^{t=2} f\,ds + \int_{t=2}^{t=3} f\,ds.$$

In each case, $\mathbf{x}'(t)$ has only one nonzero component, so the arclength differential is easy to calculate:

$$\|\mathbf{x}'(t)\| = \begin{cases} 2 & \text{if } 0 < t < 1, \\ 3 & \text{if } 1 < t < 2, \\ 2 & \text{if } 2 < t < 3. \end{cases}$$

Now we can put everything in terms of t and integrate:

$$\begin{aligned} \int_{\mathbf{x}} f\,ds &= \int_0^1 (2t+0+0)\,2\,dt + \int_1^2 (2+3t-3+0)\,3\,dt + \int_2^3 (2+3+2t-4)\,2\,dt \\ &= 2\int_0^1 2t\,dt + 3\int_1^2 3t-1\,dt + 2\int_2^3 2t+1\,dt \\ &= 2t^2\Big|_0^1 + 3\left(\tfrac{3}{2}t^2 - t\right)\Big|_1^2 + 2\left(t^2+t\right)\Big|_2^3 = 2 + \tfrac{21}{2} + 12 = \tfrac{49}{2}. \end{aligned}$$

22. Recall that you can calculate the work done by a force field on a moving particle by integrating the force field over the path that the particle traverses. So we need to find the value of

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} x^2y\,dx + z\,dy + (2x-y)\,dz$$

where $\mathbf{x}$ is the straight line from $(1,\,1,\,1)$ to $(2,\,-3,\,3)$. It has the parameterization $\mathbf{x}(t) = (1+t,\,1-4t,\,1+2t)$ where t goes from 0 to 1, and taking the derivative gives $\mathbf{x}'(t) = (1,\,-4,\,2)$. To finish up, substitute and integrate:

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} &= \int_0^1 (1+t)^2(1-4t)\,dt + (1+2t)(-4)\,dt + (2(1+t)-(1-4t))\,2\,dt \\ &= \int_0^1 \left(-4t^3 - 7t^2 - 2t + 1\right) + (-8t-4) + (12t+2)\,dt \\ &= \int_0^1 -4t^3 - 7t^2 + 2t - 1\,dt \\ &= -t^4 - \tfrac{7}{3}t^3 + t^2 - t\Big|_0^1 = -\tfrac{10}{3}. \end{aligned}$$

34. The area of one side of the fence is given by

$$\int_{\mathbf{x}} h\,ds.$$

We can parameterize the quarter circle by $\mathbf{x}(t) = (5\cos t,\,5\sin t)$ where t goes from 0 to $\frac{\pi}{2}$. It has derivative $\mathbf{x}'(t) = (-5\sin t,\,5\cos t)$. To do the integral, first find the arclength differential

$$\begin{aligned} ds &= \|\mathbf{x}'(t)\|\,dt \\ &= \sqrt{(-5\sin t)^2 + (5\cos t)^2}\,dt \\ &= \sqrt{25\sin^2 t + 25\cos^2 t}\,dt \\ &= \sqrt{25}\,dt = 5\,dt. \end{aligned}$$

The value of the integral is

$$\begin{aligned} \int_{\mathbf{x}} h\,ds &= \int_0^{\pi/2} (10 - 5\cos t - 5\sin t)\,5\,dt \\ &= 25\int_0^{\pi/2} 2 - \cos t - \sin t\,dt \\ &= 25\,(2t - \sin t + \cos t)\Big|_0^{\pi/2} = 25\pi - 50. \end{aligned}$$

38. Let's take the book's hint and take the derivative of the equation for the sphere (using the chain rule).

$$\begin{aligned} x^2 + y^2 + z^2 &= c^2 \\ \frac{d}{dt}\left(x^2 + y^2 + z^2\right) &= \frac{d}{dt}c^2 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} &= 0 \end{aligned}$$

If $x(t)$, $y(t)$, and $z(t)$ are the coordinate functions of our path, $\mathbf{x}(t)$, then what this tells us is that

$$2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$$

for any t. Now if we try to integrate $\mathbf{F}$ we get

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{x}} x\,dx + y\,dy + z\,dz \\ &= \int x(t)x'(t)\,dt + y(t)y'(t)\,dt + z(t)z'(t)\,dt. \end{aligned}$$

If $2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$ then we can certainly divide by two to see that $x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = 0$. As the integrand in the integral above is 0, we can conclude that $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = 0$.

6.2

6. First, let's calculate the line integral part of Green's theorem. We can use our usual parameterization for a circle: $\mathbf{x}(t) = (r\cos t, r\sin t)$ where r is the radius of the circle and t goes from 0 to 2π. The derivative is $\mathbf{x}'(t) = (-r\sin t, r\cos t)$. Since we want to traverse the boundary of D with the region on the left, we need to go counterclockwise around the outer circle and clockwise around the inner. Recall that integrating

clockwise over a curve gives you minus the value of the counterclockwise integral, so we want to find

$$\oint_{C_3} \mathbf{F}\cdot d\mathbf{s} - \oint_{C_2} \mathbf{F}\cdot d\mathbf{s},$$

where C_r is the circle with radius r, oriented counterclockwise. We integrate

$$\begin{aligned}
\oint_{C_3} \mathbf{F}\cdot d\mathbf{s} &= \oint_{C_3} \left(x^2y + x\right) dx + \left(y^3 - xy^2\right) dy \\
&= \int_0^{2\pi} \left(-81\sin^2 t\cos^2 t - 9\sin t\cos t\right) dt + \left(81\sin^3 t\cos t - 81\sin^2 t\cos^2 t\right) dt \\
&= -162\int_0^{2\pi} \sin^2 t\cos^2 t\, dt - 9\int_0^{2\pi} \sin t\cos t\, dt + 81\int_0^{2\pi} \sin^3 t\cos t\, dt \\
&= -\tfrac{81}{2}\int_0^{2\pi} \sin^2 2t\, dt - 9\sin^2 t\Big|_0^{2\pi} + 81\sin^4 t\Big|_0^{2\pi} \\
&= -\tfrac{81}{4}\int_0^{2\pi} 1 - \cos 4t\, dt - 9\sin^2 t\Big|_0^{2\pi} + 81\sin^4 t\Big|_0^{2\pi} \\
&= -\tfrac{81}{4}\left(t - \tfrac{1}{4}\sin 4t\right)\Big|_0^{2\pi} - 9\sin^2 t\Big|_0^{2\pi} + 81\sin^4 t\Big|_0^{2\pi} = -\tfrac{81}{2}\pi
\end{aligned}$$

and similarly

$$\begin{aligned}
\oint_{C_2} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi} \left(-16\sin^2 t\cos^2 t - 4\sin t\cos t\right) dt + \left(16\sin^3 t\cos t - 16\sin^2 t\cos^2 t\right) dt \\
&= -32\int_0^{2\pi} \sin^2 t\cos^2 t\, dt - 4\int_0^{2\pi} \sin t\cos t\, dt + 16\int_0^{2\pi} \sin^3 t\cos t\, dt \\
&= -4\int_0^{2\pi} 1 - \cos 4t\, dt - 4\int_0^{2\pi} \sin t\cos t\, dt + 16\int_0^{2\pi} \sin^3 t\cos t\, dt \\
&= -4\left(t - \tfrac{1}{4}\sin 4t\right) - 4\sin^2 t + 16\sin^4 t\Big|_0^{2\pi} = -8\pi.
\end{aligned}$$

So the value of the entire integral is $-\frac{81}{2}\pi - (-8\pi) = -\frac{65}{2}\pi$.

Now for the double integral. We want to calculate

$$\iint_D \left(\frac{\partial}{\partial x}\left(y^3 - xy^2\right) - \frac{\partial}{\partial y}\left(x^2y + x\right)\right) dA = \iint_D \left(-y^2 - x^2\right) dA.$$

Polar coordinates help a lot with this one because a) it's a circular region and b) the integrand turns out to be very simple in polar coordinates. The limits of integration will be $0 \le \theta \le 2\pi$ and $2 \le r \le 3$. Remember that in polar coordinates $dA = r\,dr\,d\theta$.

$$\begin{aligned}
-\iint_D x^2 + y^2\, dA &= -\int_0^{2\pi}\int_2^3 r^2\, r\, dr\, d\theta \\
&= -2\pi\left(\tfrac{1}{4}r^4\right)\Big|_2^3 \\
&= -2\pi\left(\tfrac{81}{4} - \tfrac{16}{4}\right) = -\tfrac{65}{2}\pi
\end{aligned}$$

16. As you will see in problem 17, it is a consequence of Green's theorem that you can find the area of a plane region using either of the integrals

$$\text{Area of } D = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx.$$

Let's use the first one. Similar to the previous problem, we need to integrate

$$\oint_C x\,dy - \oint_E x\,dy,$$

where C is the circle and E is the ellipse. We'll use the usual parameterization for the circle with radius 5: $\mathbf{x}(t) = (5\cos t,\, 5\sin t)$ and $\mathbf{x}'(t) = (-5\sin t,\, 5\cos t)$ where t goes from 0 to 2π. An ellipse is similar to a circle except that the coefficients of sin and cos are different: $\mathbf{y}(t) = (3\cos t,\, 2\sin t)$ so $\mathbf{y}'(t) = (-3\sin t,\, 2\cos t)$ and again t goes from 0 to 2π. Integrating over the circle gives us

$$\begin{aligned}\oint_C x\,dy &= \int_0^{2\pi} (5\cos t)(\,5\cos t)\,dt \\ &= 25\int_0^{2\pi} \cos^2 t\,dt \\ &= \tfrac{25}{2}\int_0^{2\pi} 1+\cos 2t\,dt \\ &= \tfrac{25}{2}\left(t+\tfrac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = 25\pi\end{aligned}$$

and over the ellipse is

$$\begin{aligned}\oint_E x\,dy &= \int_0^{2\pi} (3\cos t)(\,2\cos t)\,dt \\ &= 6\int_0^{2\pi} \cos^2 t\,dt \\ &= 3\left(t+\tfrac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = 6\pi.\end{aligned}$$

So the area of the region is $25\pi - 6\pi = 19\pi$.

6.3

6. Since the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$, we can check whether it is conservative by seeing if $\nabla\times\mathbf{F} = 0$. As $\mathbf{F}$ is a two-dimensional vector field, its curl is

$$\begin{aligned}\nabla\times\mathbf{F} &= \left(\frac{\partial}{\partial x}\left(\frac{x^2y}{1+x^2}\right) - \frac{\partial}{\partial y}\left(\frac{xy^2}{(1+x^2)^2}\right)\right)\mathbf{k} \\ &= \left(\frac{2xy(1+x^2)-2x^3y}{(1+x^2)^2} - \frac{2xy}{(1+x^2)^2}\right)\mathbf{k} \\ &= \frac{2xy+2x^3y-2x^3y-2xy}{(1+x^2)^2}\mathbf{k} = 0.\end{aligned}$$

The curl of **F** is 0, which means that it is a conservative vector field. If f is a scalar potential function for **F** then we will need

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{xy^2}{(1+x^2)^2} \text{ and}\\ \frac{\partial f}{\partial y} &= \frac{x^2 y}{1+x^2}.\end{aligned}$$

Let's integrate $\frac{\partial f}{\partial y}$ with respect to y, since it looks easier than integrating $\frac{\partial f}{\partial x}$ with respect to x.

$$\begin{aligned}f(x,y) = \int \frac{\partial f}{\partial y}\,dy &= \int \frac{x^2 y}{1+x^2}\,dy\\ &= \frac{x^2}{1+x^2}\int y\,dy\\ &= \frac{x^2}{1+x^2}\left(\tfrac{1}{2}y^2 + g(x)\right),\end{aligned}$$

where g is some function depending on x but not on y. Now differentiate our expression for $f(x,y)$ with respect to x:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{2x(1+x^2)-2x^3}{(1+x^2)^2}\left(\tfrac{1}{2}y^2 + g(x)\right) + \frac{x^2}{1+x^2}g'(x)\\ &= \frac{2x}{(1+x^2)^2}\left(\tfrac{1}{2}y^2 + g(x)\right) + \frac{x^2}{1+x^2}g'(x)\\ &= \frac{xy^2}{(1+x^2)^2} + \frac{x}{1+x^2}\left(\frac{2}{1+x^2}g(x) + xg'(x)\right).\end{aligned}$$

We want $\frac{\partial f}{\partial x}$ to be equal to the first coordinate function of **F**, which is $\frac{xy^2}{(1+x^2)^2}$. Thus, we need to choose g so that

$$\frac{2}{1+x^2}g(x) + xg'(x) = 0.$$

An obvious choice is $g(x) = 0$. This makes our potential function

$$f(x,y) = \frac{x^2}{1+x^2}\left(\tfrac{1}{2}y^2\right) = \frac{x^2y^2}{2(1+x^2)}.$$

20. The vector field **F** will be conservative if $\nabla \times \mathbf{F} = 0$. We can calculate the curl of **F**:

$$\nabla \times \mathbf{F} = \left((\sin y + y\sin x) - \frac{\partial M}{\partial y}\right)\mathbf{k},$$

so **F** will have curl 0 if

$$\frac{\partial M}{\partial y} = \sin y + y \sin x.$$

Find M by integrating this expression with respect to y.

$$\begin{aligned} M(x,y) = \int \frac{\partial M}{\partial y}\,dy &= \int \sin y + y \sin x \, dy \\ &= -\cos y + \tfrac{1}{2}y^2 \sin x + f(x), \end{aligned}$$

where f is any (C^1) function depending on x but not y.

26. The integral will be path independent if the integrand is a conservative vector field. Our vector field is $\mathbf{F}(x,y) = (3x-5y)\mathbf{i} + (7y-5x)\mathbf{j}$ and its curl is

$$\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}(7y-5x) - \frac{\partial}{\partial y}(3x-5y) \right)\mathbf{k} = (-5-(-5))\,\mathbf{k} = 0,$$

so the integral is indeed path independent. We can evaluate it directly by parameterizing the path as $\mathbf{x}(t) = (1+4t)\mathbf{i} + (3-t)\mathbf{j}$ where t goes from 0 to 1. Then $\mathbf{x}'(t) = 4\mathbf{i} - \mathbf{j}$, and we can integrate

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (3(1+4t) - 5(3-t))\,4\,dt + (7(3-t) - 5(1+4t))\,(-1)\,dt \\ &= \int_0^1 95t - 64\,dt = \tfrac{95}{2}t^2 - 64t\Big|_0^1 = -\tfrac{33}{2}. \end{aligned}$$

The other way we can evaluate the integral is by computing $f(5,2) - f(1,3)$, where f is a scalar potential function for $\mathbf{F}$. We want to have

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x - 5y \text{ and} \\ \frac{\partial f}{\partial y} &= 7y - 5x. \end{aligned}$$

Integrating $\frac{\partial f}{\partial x}$ with respect to x gives

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int 3x - 5y\,dx = \tfrac{3}{2}x^2 - 5xy + g(y),$$

for a C^1 function g. Now let's check $\frac{\partial f}{\partial y}$.

$$7y - 5x = \frac{\partial f}{\partial y} = -5x + g'(y)$$

We need $g'(y) = 7y$, so let's set $g(y) = \frac{7}{2}y^2$. Our potential function turns out to be

$$f(x,y) = \tfrac{3}{2}x^2 - 5xy + \tfrac{7}{2}y^2.$$

Finally, we can calculate

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= f(5,2) - f(1,3) \\ &= \tfrac{3}{2}(25) - 5(10) + \tfrac{7}{2}(4) - \left(\tfrac{3}{2} - 5(3) + \tfrac{7}{2}(9)\right) = -\tfrac{33}{2}. \end{aligned}$$

7.1

4. (a) Our parameterized surface is $\mathbf{X}(s,t) = (s^2\cos t,\ s^2\sin t,\ s)$. Its partial derivatives are

$$\frac{\partial \mathbf{X}}{\partial s} = (2s\cos t,\ 2s\sin t,\ 1) \text{ and}$$
$$\frac{\partial \mathbf{X}}{\partial t} = \left(-s^2\sin t,\ s^2\cos t,\ 0\right).$$

We can find a normal vector to the surface by taking the cross product of its tangent vectors.

$$\begin{aligned}\mathbf{N}(-1,\ 0) &= \left.\frac{\partial \mathbf{X}}{\partial s}\right|_{(-1,0)} \times \left.\frac{\partial \mathbf{X}}{\partial t}\right|_{(-1,0)} \\ &= (-2\cos 0,\ -2\sin 0,\ 1) \times (-(-1)^2\sin 0,\ (-1)^2\cos 0,\ 0) \\ &= (-2,\ 0,\ 1) \times (0,\ 1,\ 0) = (-1,\ 0,\ -2).\end{aligned}$$

(b) The tangent plane at $(-1,\ 0)$ consists of points $\mathbf{x}$ for which

$$\mathbf{N}(-1,0)\cdot(\mathbf{x} - \mathbf{X}(-1,0)) = 0.$$

First, calculate $\mathbf{X}(-1,0) = (1,\ 0,\ -1)$. Now, if $\mathbf{x} = (x,\ y,\ z)$ then our equation is

$$\begin{aligned}0 &= (-1,\ 0,\ -2)\cdot(x-1,\ y,\ z+1) \\ &= -(x-1) - 2(z+1) \\ &= -x - 2z - 1.\end{aligned}$$

(c) Here we'll use x, y, and z for the coordinate functions of our surface:

$$\begin{aligned}x(t) &= s^2\cos t, \\ y(t) &= s^2\sin t, \text{ and} \\ z(t) &= s.\end{aligned}$$

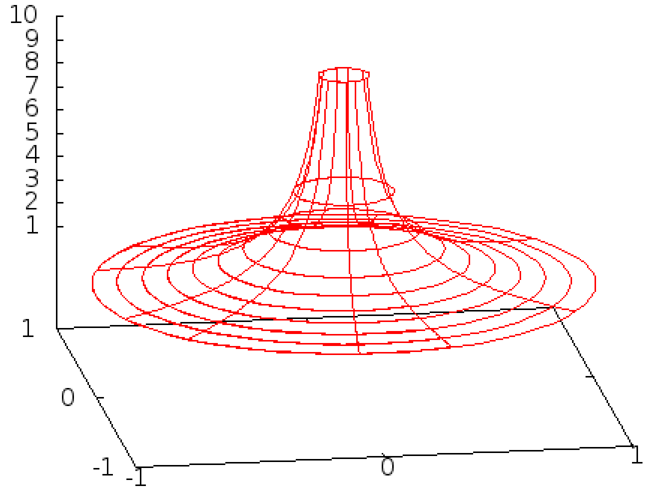
To get rid of t, recall that $\sin^2 t + \cos^2 = 1$, so we can do

$$[x(t)]^2 + [y(t)]^2 = sin^4\cos^2 t + s^4\sin^2 t = s^4.$$

But the right hand side is just $[z(t)]^4$, so the equation that defines this surface is

$$x^2 + y^2 = z^4.$$

30. (a) To sketch this surface, it helps to translate it into cylindrical coordinates. This results in $z = \frac{1}{r}$, so this surface is what we get by revolving $z = \frac{1}{x}$, for $x \geq 1$, around the z-axis. It looks like this (z is on the vertical axis):



(b) As you can see from the graph, the region bounded by the surface and the plane $z = 1$ is the part with $r \le 1$. An improper integral is needed to calculate this volume because z goes to infinity as r goes to 0. The volume is

$$\begin{aligned}
\int_0^{2\pi}\int_0^1 \left(\frac{1}{r} - 1\right) r\,dr\,d\theta &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1 \left(\frac{1}{r} - 1\right) r\,dr\,d\theta \\
&= 2\pi \lim_{\epsilon\to 0}\int_\epsilon^1 1 - r\,dr \\
&= 2\pi \lim_{\epsilon\to 0} r - \tfrac{1}{2}r^2\Big|_\epsilon^1 \\
&= 2\pi \lim_{\epsilon\to 0} 1 - \tfrac{1}{2} - \left(\epsilon - \tfrac{1}{2}\epsilon^2\right) \\
&= 2\pi \lim_{\epsilon\to 0} \tfrac{1}{2} - \epsilon - \frac{\epsilon^2}{2} \\
&= 2\pi\left(\tfrac{1}{2} - 0 - \tfrac{0}{2}\right) = \pi.
\end{aligned}$$

(c) If we parameterize the surface by

$$\mathbf{X}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ \tfrac{1}{r})$$

then the two tangent vectors are

$$\begin{aligned}
\mathbf{T}_r(r,\ \theta) &= (\cos\theta,\ \sin\theta,\ -r^{-2}) \text{ and} \\
\mathbf{T}_\theta(r,\ \theta) &= (-r\sin\theta,\ r\cos\theta,\ 0),
\end{aligned}$$

so a normal vector is

$$\mathbf{N}(r,\theta) = \mathbf{T}_r(r,\ \theta) \times \mathbf{T}_\theta(r,\ \theta) = \left(\tfrac{1}{r}\cos\theta,\ \tfrac{1}{r}\sin\theta,\ r\right)$$

and its magnitude is

$$\|\mathbf{N}(r,\theta)\| = \sqrt{\left(\tfrac{1}{r}\cos\theta\right)^2 + \left(\tfrac{1}{r}\sin\theta\right)^2 + r^2} = \sqrt{\frac{1}{r^2} + r^2}.$$

Now this might look a little scary to integrate, but we can simplify things a little first. Since r^2 is positive,

$$\sqrt{\frac{1}{r^2} + r^2} \geq \sqrt{\frac{1}{r^2}} = \frac{1}{r}.$$

The inequality extends to the integral, so the surface area is

$$\begin{aligned}
\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{r^2} + r^2}\, dr\, d\theta &\geq \int_0^{2\pi}\int_0^1 \frac{1}{r}\, dr\, d\theta \\
&= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1 \frac{1}{r}\, dr\, d\theta \\
&= 2\pi \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{r}\, dr \\
&= 2\pi \lim_{\epsilon\to 0} \log r\Big|_\epsilon^1 \\
&= 2\pi \lim_{\epsilon\to 0} \log 1 - \log\epsilon = \infty.
\end{aligned}$$

This last limit is ∞ because $\log\epsilon$ goes to $-\infty$ as ϵ goes to 0. Since the surface area is greater than an infinite quantity, it too is infinite.

Another way to think about this part would be to use the description of the surface we gave earlier—it is what you get by revolving the part of $z = \frac{1}{x}$ with $x > 0$ around the z-axis. The area of this surface of revolution is $\int 2\pi x\, ds$, where ds is the arclength differential $ds = \sqrt{1 + \left(\frac{dz}{dx}\right)^2}\, dx$. This actually results in the same calculation since $\frac{dz}{dx} = -x^{-2}$ and

$$\int 2\pi x\sqrt{1 + (-x^{-2})^2}\, dx = 2\pi \int \sqrt{x^2 + x^{-2}}\, dx.$$