3.1

T/F

2. T

4. T

Problems

10.

$$\begin{vmatrix} -4 & 10 \\ -1 & 8 \end{vmatrix} = (-4)(8)) - (-1)(10) = -32 + 10 = -22$$

18.

$$\begin{aligned} \begin{vmatrix} 3 & 2 & 6 \\ 2 & 1 & -1 \\ -1 & 1 & 4 \end{vmatrix} &= \begin{aligned} &(3)(1)(4) + (2)(-1)(-1) + (6)(2)(1) \\ &- (-1)(1)(6) - (1)(-1)(3) - (4)(2)(2) \end{aligned} \\ &= 12 + 2 + 12 + 6 + 3 - 16 \\ &= 19 \end{aligned}$$

22. If

$$\begin{aligned} y_1(x) &= \cos 2x, \\ y_2(x) &= \sin 2x, \text{ and} \\ y_3(x) &= e^x \end{aligned}$$

then

$$\begin{gathered} y_1''' - y_1'' + 4y_1' - 4y_1 = 8\sin 2x + 4\cos 2x - 8\sin 2x - 4\cos 2x = 0, \\ y_2''' - y_2'' + 4y_2' - 4y_2 = -8\cos 2x + 4\sin 2x + 8\cos 2x - 4\sin 2x = 0, \text{ and} \\ y_3''' - y_3'' + 4y_3' - 4y_3 = e^x - e^x + 4e^x - 4e^x. \end{gathered}$$

The Wronskian is

$$\begin{aligned} \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} &= \begin{vmatrix} \cos 2x & \sin 2x & e^x \\ -2\sin 2x & 2\cos 2x & e^x \\ -4\cos 2x & -4\sin 2x & e^x \end{vmatrix} \\ &= \begin{aligned} &2e^x\cos^2 2x - 4e^x \sin 2x \cos 2x + 8e^x \sin^x 2x \\ &+ 8e^x \cos^2 2x + 4e^x \sin 2x \cos 2x + 2e^x \sin^2 2x \end{aligned} \\ &= 2e^x + 8e^x. \end{aligned}$$

This function is positive for all real x because e^x is positive.

24. (a,b) We can list the permutations in such a way that neighboring ones differ only by a transposition. This means that the parity will alternate between even and odd.

Permutation	**Parity**
$(1,2,3,4)$	even
$(1,2,4,3)$	odd
$(1,4,2,3)$	even
$(4,1,2,3)$	odd
$(4,1,3,2)$	even
$(1,4,3,2)$	odd
$(1,3,4,2)$	even
$(1,3,2,4)$	odd
$(3,1,2,4)$	even
$(3,1,4,2)$	odd
$(3,4,1,2)$	even
$(4,3,1,2)$	odd

Permutation	**Parity**
$(4,3,2,1)$	even
$(3,4,2,1)$	odd
$(3,2,4,1)$	even
$(3,2,1,4)$	odd
$(2,3,1,4)$	even
$(2,3,4,1)$	odd
$(2,4,3,1)$	even
$(4,2,3,1)$	odd
$(4,2,1,3)$	even
$(2,4,1,3)$	odd
$(2,1,4,3)$	even
$(2,1,3,4)$	odd

(c) The forumula for the determinant of a 4×4 matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{aligned} & a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} + a_{11}a_{24}a_{32}a_{43} - a_{14}a_{21}a_{32}a_{43} \\ & + a_{14}a_{21}a_{33}a_{42} - a_{11}a_{24}a_{33}a_{42} + a_{11}a_{23}a_{34}a_{42} - a_{11}a_{23}a_{32}a_{44} \\ & + a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} + a_{13}a_{24}a_{31}a_{42} - a_{14}a_{23}a_{31}a_{42} \\ & + a_{14}a_{23}a_{32}a_{41} - a_{13}a_{24}a_{32}a_{41} + a_{13}a_{22}a_{34}a_{41} - a_{13}a_{22}a_{31}a_{44} \\ & + a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41} + a_{12}a_{24}a_{33}a_{41} - a_{14}a_{22}a_{33}a_{41} \\ & + a_{14}a_{22}a_{31}a_{43} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{12}a_{21}a_{33}a_{44} \end{aligned}$$

3.2

T/F

2. T

Problems

2.

$$\begin{aligned} \begin{vmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 1 & 4 \end{vmatrix} &= \begin{vmatrix} 2 & -1 & 4 \\ 0 & \frac{7}{2} & -5 \\ -2 & 1 & 4 \end{vmatrix} && A_{12}\left(-\tfrac{3}{2}\right) \\ &= \begin{vmatrix} 2 & -1 & 4 \\ 0 & \frac{7}{2} & -5 \\ 0 & 0 & 8 \end{vmatrix} && A_{13}\,(1) \\ &= (2)\left(\tfrac{7}{2}\right)(8) = 56 \end{aligned}$$

Once the matrix is in upper triangular form, its determinant can be immediately evaluated as the product of the elements on the main diagonal.

16. If

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{bmatrix}$$

then Theorem 3.2.4 says that A is invertible if and only if $\det(A) \neq 0$. We calculate

$$\begin{aligned}
\det(A) = \begin{vmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{vmatrix} &= -\begin{vmatrix} 1 & -2 & -3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{vmatrix} && M_1(-1) \\
&= -\begin{vmatrix} 1 & -2 & -3 \\ 0 & 8 & 16 \\ 0 & 14 & 29 \end{vmatrix} && A_{12}(-5), A_{13}(-8) \\
&= -8\begin{vmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 14 & 29 \end{vmatrix} && M_2\left(\tfrac{1}{8}\right) \\
&= -8\begin{vmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} && A_{23}(-14) \\
&= -8.
\end{aligned}$$

Since the determinant is nonzero, A is invertible.

36. Use properties of determinants:

$$\det\left(AB^T\right) = \det(A)\det\left(B^T\right) = \det(A)\det(B) = (5)(3) = 15.$$

54. Skew-symmetric means that $A^T = -A$. We use properties of the determinant:

$$\det(A) = \det\left(A^T\right) = \det(-A) = -\det(A).$$

Since $\det(A) = -\det(A)$, it must be the case that $\det(A) = 0$.

4.2

T/F

6. T

Problems

6. The set S is the set of singular 2×2 matrices with real entries.

   (a) The zero vector in $M_2(\mathbb{R})$ is the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We check that

   $$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

   which tells us that this matrix is in S.

(b) Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The two summands are in S because

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix},$$

but the right hand side is not in S because

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

(c) Yes, S is closed under scalar multiplication. If $c \in \mathbb{R}$, and $A \in S$, we know from our study of determinants that

$$\det(cA) = c^2 \det(A) = c^2(0) = 0,$$

so $cA \in S$.

10. If $A = (a_{ij})_{m\times n}$ and $Z = (z_{ij})_{m\times n}$ then

$$A + Z = (a_{ij} + z_{ij})_{m\times n},$$

so $A + Z = A$ when $z_{ij} = 0$ for all i and j. Thus, the zero vector in $M_{m\times n}(\mathbb{R})$ is the zero matrix $Z = (0)_{m\times n}$, and from now on we will just write 0 instead of Z.

Let $B = (b_{ij})_{m\times n}$. Then

$$0 = A + B = (a_{ij} + b_{ij})_{m\times n}$$

if $b_{ij} = -a_{ij}$ for all i and j. With B defined in this way, B is the additive inverse of A and we can write $B = -A$.

4.3

Problems

4.

$$S = \left\{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = (x_1, 0, x_3, 2)\,,\ x_1 \in \mathbb{R},\ x_3 \in \mathbb{R}\right\}$$

In order to be a subspace, S must contain the zero vector of V. The zero vector of $\mathbb{R}^4$ is $(0, 0, 0, 0)$. We can see that this vector is not in S because the last coordinates of $(x_1, 0, x_3, 2)$ and $(0, 0, 0, 0)$ do not agree. This means that S is not a subspace of V.

16.

$$S = \left\{p(x) \in P_2 : p(x) = ax^2 + b,\ a \in \mathbb{R},\ b \in \mathbb{R}\right\}$$

First, let's check that S contains the zero vector. The zero vector in P_2 is $0x^2 + 0x + 0$. This is the element of S with $a = 0$ and $b = 0$, so $\mathbf{0} \in S$. Now we need to check that

S is closed under addition and scalar multiplication. Two generic elements of S are $ax^2 + b$ and $cx^2 + d$, with $a, b, c, d \in \mathbb{R}$.

$$\left(ax^2 + b\right) + \left(cx^2 + d\right) = (a + c)x^2 + (b + d) \in S$$

Also, if $k \in \mathbb{R}$ then

$$k\left(ax^2 + b\right) = (ka)x^2 + (kb) \in S.$$

Since S is a nonempty subset of V that is closed under addition and scalar multiplication, S is a subspace of V.

24. (a) Let's check the properties that $S_1 \cup S_2$ needs in order to be a subspace. First, we check that $\mathbf{0} \in S_1 \cup S_2$. Well, $\mathbf{0} \in S_1$ *and* $\mathbf{0} \in S_2$, so $\mathbf{0}$ is certainly in $S_1 \cup S_2$. Next, if $c \in \mathbb{R}$ and $\mathbf{x} \in S_1 \cup S_2$ then either $\mathbf{x} \in S_1$, in which case $c\mathbf{x} \in S_1$, or $\mathbf{x} \in S_2$, in which case $c\mathbf{x} \in S_2$, so $c\mathbf{x} \in S_1 \cup S_2$.

    The one property left is being closed under addition, and this is the one that fails. If $\mathbf{x} \in S_1 \cup S_2$ and $\mathbf{y} \in S_1 \cup S_2$ then there are three cases: either $\mathbf{x}$ and $\mathbf{y}$ are both in S_1, both in S_2, or one is in S_1 and the other is in S_2. If both $\mathbf{x}$ and $\mathbf{y}$ lie in the same subspace, say in S_1, then their sum is also in S_1. The problem comes when you try to add a vector in one subspace to a vector in the other. In general, there is no guarantee that the sum is in either one. Thus, $S_1 \cup S_2$ is not, in general, a subspace of V.

    (b) As noted in (a), $\mathbf{0} \in S_1$ and $\mathbf{0} \in S_2$ because both are subspaces of V, so $\mathbf{0} \in S_1 \cap S_2$. If $\mathbf{x}$ and $\mathbf{y}$ are in $S_1 \cap S_2$ then $\mathbf{x}, \mathbf{y} \in S_1$, which tells us that $\mathbf{x} + \mathbf{y} \in S_1$, and $\mathbf{x}, \mathbf{y} \in S_2$, so $\mathbf{x} + \mathbf{y} \in S_2$. Together, this tells us that $\mathbf{x} + \mathbf{y} \in S_1 \cap S_2$. Finally, if $c \in \mathbb{R}$ and $\mathbf{x} \in S_1 \cap S_2$ then $c\mathbf{x} \in S_1$ because $\mathbf{x} \in S_1$ and $c\mathbf{x} \in S_2$ because $\mathbf{x} \in S_2$. Hence, $c\mathbf{x} \in S_1 \cap S_2$, and we can conclude that $S_1 \cap S_2$ is a subspace of V.

    (c) We can write $\mathbf{0} = \mathbf{0} + \mathbf{0}$ to show that $\mathbf{0} \in S_1 + S_2$. If $\mathbf{x} + \mathbf{y} \in S_1 + S_2$ and $\mathbf{w} + \mathbf{z} \in S_1 + S_2$ with $\mathbf{x}, \mathbf{w} \in S_1$ and $\mathbf{y}, \mathbf{z} \in S_2$ then

    $$(\mathbf{x} + \mathbf{y}) + (\mathbf{w} + \mathbf{z}) = (\mathbf{x} + \mathbf{w}) + (\mathbf{y} + \mathbf{z})$$

    is in $S_1 + S_2$ because $\mathbf{x} + \mathbf{w} \in S_1$ and $\mathbf{y} + \mathbf{z} \in S_2$. Now suppose $c \in \mathbb{R}$ and $\mathbf{x} + \mathbf{y} \in S_1 + S_2$ with $\mathbf{x} \in S_1$ and $\mathbf{y} \in S_2$ then

    $$c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$$

    is in $S_1 + S_2$ because $c\mathbf{x} \in S_1$ and $c\mathbf{y} \in S_2$.

    Notice that for any $\mathbf{x} \in S_1$ we have $\mathbf{x} = \mathbf{x} + \mathbf{0} \in S_1 + S_2$ and for $\mathbf{y} \in S_2$, write $\mathbf{y} = \mathbf{0} + \mathbf{y} \in S_1 + S_2$. What this shows is that $S_1 \cup S_2$ is a subset (but not a subspace) of $S_1 + S_2$. In fact, $S_1 + S_2$ is the smallest subspace of V that contains $S_1 \cup S_2$ because we have corrected the one problem in part (a); $S_1 + S_2$ contains the sums of vectors from S_1 and S_2.

4.4

T/F

4. F

6. F

Problems

12. The definition of S involves two parameters: c_1 and c_2. We can determine a spanning set for S by "pulling out" the coefficients of c_1 and c_2. Writing the vectors as columns,

$$\begin{bmatrix} c_1 \\ c_2 \\ c_2 - c_1 \\ c_1 - 2c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

So these two vectors—$(1,0,-1,1)$ and $(0,1,1,-2)$—span the subspace S.

14. To find the null space of A, we need to find the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Do this by Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \xrightarrow{A_{12}(-3), A_{13}(-5)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}$$
$$\xrightarrow{M_2\left(-\frac{1}{2}\right)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{bmatrix}$$
$$\xrightarrow{A_{23}(4)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{A_{21}(-2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The reduced row-echelon form of A tells us that the solution set to $A\mathbf{x} = \mathbf{0}$ is

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (t, -2t, t),\ t \in \mathbb{R} \right\}.$$

A spanning set for the null space of A is given by the coefficients of the free variable t; the null space of A is spanned by $(1,-2,1)$.

16. A generic element of $M_2(\mathbb{R})$ looks like

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If the matrix is symmetric then $a_{21} = a_{12}$, so this matrix is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}.$$

Now we can write

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} &= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= a_{11}A_1 + a_{22}A_2 + a_{12}A_3 \end{aligned}$$

and this shows that A_1, A_2, and A_3 span this subspace of $M_2(\mathbb{R})$.