Final Exam Practice Problems

Math 240 — Calculus III

Summer 2013, Session II

Vector Calculus

1. Which of the following statements are true for all C^2 scalar functions $f : \mathbb{R}^3 \to \mathbb{R}$ and vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$?

   (a) $\nabla \cdot (\nabla f) = 0$

   (b) $\nabla \times (\nabla f) = \mathbf{0}$

   (c) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

   Answer: Statements (b) and (c) are true.

2. Calculate the arc length of the given curve in $\mathbb{R}^3$.

   (a) $\mathbf{x}(t) = (3\cos 2t,\ 3\sin 2t,\ 3t)$ for $0 \le t \le \frac{\pi}{2}$
   Answer: $\frac{3}{2}\pi\sqrt{5}$

   (b) $\mathbf{x}(t) = \left((t^2+1)\cos t,\ (t^2+1)\sin t,\ 2\sqrt{2}t\right)$ for $0 \le t \le 1$
   Answer: $\frac{10}{3}$

3. Compute the gradient of the given function.

   (a) $f(x, y, z) = x^2 e^{yz}$
   Answer: $2xe^{yz}\,\mathbf{i} + x^2 z e^{yz}\,\mathbf{j} + x^2 y e^{yz}\,\mathbf{k}$

   (b) $\ln(xy) + y\sin z$
   Answer: $x^{-1}\,\mathbf{i} + \left(y^{-1} + \sin z\right)\,\mathbf{j} + y\cos z\,\mathbf{k}$

4. Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point P.

   (a) $f(x, y) = xe^y$, $P = (-1, 0, 1)$
   Answer: $z = x - y + 2$

   (b) $f(x, y) = \sqrt{x^2 + y^2}$, $P = (3, 4, 5)$
   Answer: $3x + 4y - 5z = 0$

5. Find the equation of the tangent plane to the given surface at the point P.

   (a) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$, $P = \left(1, 2, \frac{2\sqrt{11}}{3}\right)$
   Answer: $\frac{1}{2}x + \frac{4}{9}y + \frac{1}{12}\sqrt{11}z = 2$

   (b) $x\sin z = y\cos z$, $P = \left(0, 1, \frac{\pi}{2}\right)$
   Answer: $x + z = \frac{\pi}{2}$

6. Calculate $\int_C f\,ds$ for the given function f and curve C.

   (a) $f(x,y) = \frac{xy}{x^2+1}$ and C is the curve parameterized by $\mathbf{x}(t) = (t,3)$ for $0 \le t \le 2$
   Answer: $\frac{3}{2}\ln 5$

   (b) $f(x,y) = x^2 + y$ and C is the path from $(2,0)$ counterclockwise along the circle $x^2+y^2 = 4$ to the point $(-2,0)$ and then back to $(2,0)$ along the x-axis
   Answer: $4\pi + \frac{34}{3}$

   (c) $f(x,y,z) = xyz$ and C is the straight line from $(1,0,2)$ to $(-3,2,1)$
   Answer: $-2\sqrt{21}$

7. Find the lateral surface area of the part of the cylinder $x^2+y^2 = 4$ below the plane $x+2y+z = 6$ and above the xy-plane.

   Answer: 24π

8. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{s}$ for the given vector field $\mathbf{F}$ and curve C.

   (a) $\mathbf{F}(x,y) = y\,\mathbf{i} - x\,\mathbf{j}$ and C is the circle centered at the origin with radius 1, oriented clockwise.
   Answer: 2π

   (b) $\mathbf{F}(x,y) = (x^2y - \frac{1}{2}y)\,\mathbf{i} + (2x + xy^2)\,\mathbf{j}$ and C is the path parameterized by $\mathbf{x}(t) = (\cos t,\, \sin t)$ for $0 \le t \le 2\pi$
   Answer: $\frac{5}{2}\pi$ (Hint: use an indirect method.)

   (c) $\mathbf{F}(x,y) = xy^2\,\mathbf{i} + xy^3\,\mathbf{j}$ and C is the triangular path from $(0,0)$ to $(1,0)$ to $(0,1)$ and back to $(0,0)$.
   Answer: $-\frac{1}{30}$

   (d) $\mathbf{F}(x,y) = \left(x^2+y^2\right)\mathbf{i} + 2xy\,\mathbf{j}$ and C is parameterized by $\mathbf{x}(t) = (\cos t,\, \sin t + \sin 3t)$ for $0 \le t \le \pi$.
   Answer: $-\frac{2}{3}$ (Hint: indirect method for this one too.)

   (e) $\mathbf{F} = xy\,\mathbf{i} + 2z\,\mathbf{j} + 3y\,\mathbf{k}$ and C is the intersection of the cylinder $x^2+y^2 = 9$ with the plane $x + z = 5$, oriented counterclockwise from above
   Answer: 9π

9. Find a scalar potential for the given vector field, or show that none exists.

   (a) $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$
   Answer: This vector field is not conservative. $\nabla\times\mathbf{F} = -2\,\mathbf{k}$.

   (b) $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j}$
   Answer: $\mathbf{F} = \nabla\left(\frac{1}{2}x^2 - \frac{1}{2}y^2\right)$

   (c) $\mathbf{F} = x^2z^2\,\mathbf{i} + 5y\,\mathbf{j} + x^3z\,\mathbf{k}$
   Answer: $\mathbf{F}$ is not conservative. $\nabla\times\mathbf{F} = -x^2z\,\mathbf{j}$.

   (d) $\mathbf{F} = 3x^2z\,\mathbf{i} - \ln z\,\mathbf{j} + \left(x^3 - \frac{y}{z}\right)\mathbf{k}$
   Answer: $\mathbf{F} = \nabla\left(x^3z - y\ln z\right)$

10. Evaluate the given surface integral.

   (a) $\int_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$ and S is the sphere $x^2+y^2+z^2 = 9$ with outward-pointing normal
   Answer: 216π

(b) $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = z\,\mathbf{i} + 2x\,\mathbf{j} + 3y\,\mathbf{k}$ and S is the upper hemisphere ($z \geq 0$) of $x^2 + y^2 + z^2 = 9$ with upward-pointing normal

Answer: 18π

(c) $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (x^2 + z)\,\mathbf{i} - \frac{1}{3}y^3\,\mathbf{j} + \left(2x + \frac{1}{2}z^2\right)\mathbf{k}$ and S is the surface of the cube $[0,1]^3$ with outward-pointing normal

Answer: $\frac{7}{3}$

(d) $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + \mathbf{k}$ and S is the surface of the cone $z = 1 - \sqrt{x^2 + y^2}$ between $z = 0$ and $z = 1$ with normal vector pointing away from the z-axis

Answer: $-\pi$

Linear Algebra

1. Find the solution set of the given linear system.

(a) $$\begin{aligned} -5x_1 - 4x_2 &= 0 \\ -8x_1 + x_2 &= 0 \end{aligned}$$

Answer: $\{(0,0) \in \mathbb{R}^2\}$

(b) $$\begin{aligned} -x_1 + 6x_2 - 25x_3 &= 0 \\ 9x_1 + 6x_2 - 15x_3 &= 0 \end{aligned}$$

Answer: $\{(-t, 4t, t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$

(c) $$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned}$$

Answer: $\{(1,2,4) \in \mathbb{R}^3\}$

(d) $$\begin{aligned} x_1 + x_2 + 3x_3 &= 3 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + 8x_3 &= 4 \end{aligned}$$

Answer: This system is inconsistent.

(e) $$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 1 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 &= -5 \\ -x_1 - 2x_2 - x_3 - 7x_4 &= 3 \end{aligned}$$

Answer: $\{(-1 - 2s - 4t, s, -2 - 3t, t) \in \mathbb{R}^4 : s, t \in \mathbb{R}\}$ (Answers may vary.)

2. Parameterize the line that is the intersection of the planes $x + y + 3z = 4$ and $x + 2y + 4z = 5$.

Answer: $\mathbf{x}(t) = (3 - 2t,\ 1 - t,\ t)$

3. Calculate the determinant of the given matrix.

(a) $$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

Answer: -7

(b) $$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 5 \\ 4 & -1 & 3 \end{bmatrix}$$

Answer: 29

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 8 \\ 1 & 1 & 2 & 3 \\ 1 & 3 & 5 & 8 \end{bmatrix}$

Answer: -2

4. Determine whether the given matrix is invertible. Find its inverse if it has one.

(a) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Answer: The inverse of this matrix is $\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 4 & 7 & 4 \end{bmatrix}$

Answer: This matrix is not invertible.

(c) $\begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -1 \\ -1 & 0 & -3 \end{bmatrix}$

Answer: The inverse of this matrix is $\begin{bmatrix} -3 & 0 & -4 \\ -2 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$.

5. Find a basis for the subset of $\mathbb{R}^n$ spanned by the given vectors.

(a) $\mathbf{v}_1 = (1, 2, 1, 3),\ \mathbf{v}_2 = (3, 6, 3, 9),\ \mathbf{v}_3 = (1, 3, 5, 4),\ \mathbf{v}_4 = (2, 3, -2, 5)$

Answer: Answers may vary. Correct answers include

$$\{\mathbf{v}_1,\ \mathbf{v}_3\} \quad \text{and} \quad \{(1, 2, 1, 3),\ (0, 1, 4, 1)\}.$$

(b) $\mathbf{v}_1 = (1, 1, 1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 2, 4, 1)$, $\mathbf{v}_3 = (0, 0, 1, 3, 0)$, $\mathbf{v}_4 = (0, 0, 1, 4, 0)$

Answer: Answers may vary. Correct answers include

$$\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4\} \quad \text{and} \quad \{(1, 1, 1, 1, 1),\ (0, 0, 1, 3, 0),\ (0, 0, 0, 1, 0)\}.$$

6. Let $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, 2)$. Verify that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$ and express $(2, -1)$ in this basis.

Answer: One method of verification is to compute

$$\det\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\right) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

and note that it is nonzero. Then

$$(2, -1) = 5\mathbf{v}_1 - 3\mathbf{v}_2.$$

7. Determine a basis for the kernel and range of the linear transformation $T(\mathbf{v}) = A\mathbf{v}$ where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -5 \\ 3 & -2 & 8 \end{bmatrix}.$$

Answer: A basis for $\text{Ker}(T)$ is $\{(-2, 1, 1)\}$. A basis for $\text{Rng}(T)$ is $\{(1, -2, 3), (0, 1, -2)\}$. (Answers may vary.)

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(a, b) = (a - 2b, 3a)$. Find the matrix representation of T relative to the given ordered basis.

   (a) the standard basis $\{(1, 0), (0, 1)\}$

   Answer: $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$

   (b) $\{(1, 2), (1, 3)\}$

   Answer: $\begin{bmatrix} -12 & -18 \\ 9 & 13 \end{bmatrix}$

9. Let V be the subspace of $C^\infty(\mathbb{R})$ spanned by $y_1 = e^{2x} \cos x$ and $y_2 = e^{2x} \sin x$. Find the matrix representation of the linear transformation $T : V \to V$ given by $T(f) = f' + 3f$ relative to the ordered basis $\{y_1, y_2\}$.

   Answer: $\begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}$

10. Determine whether the statement is true or false.

   (a) The set of invertible $n \times n$ matrices is a subspace of $M_n(\mathbb{R})$.

   Answer: False.

   (b) The set $\{(a, b, 0, a) \in \mathbb{R}^4 : a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$.

   Answer: True.

   (c) The mapping $T : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ defined by $T(f) = f'' - 3f' + 5f$ is a linear transformation.

   Answer: True.

   (d) If the standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are eigenvectors of an $n \times n$ matrix, then the matrix is diagonal.

   Answer: True.

   (e) If 1 is the only eigenvalue of an $n \times n$ matrix, then it must be the identity matrix.

   Answer: False.