Math 114 Review

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 1, 2013

Agenda

1. Gradient, Divergence, and Curl
   - Gradient
   - Divergence
   - Curl
   - How they're related
2. Line integrals
   - Scalar line integrals
   - Vector line integrals
   - Conservative vector fields

Gradient

Definition

Let $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a differentiable scalar function on a region of 3-dimensional space. The **gradient** of f is the vector field

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The direction of the gradient, $\frac{\nabla f}{\|\nabla f\|}$, is the direction in which f is increasing the fastest. The norm, $\|\nabla f\|$, is the rate of this increase.

Example

If $f(x, y, z) = x^2 + y^2 + z^2$ then

$$\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

Divergence

Definition

Let $\mathbf{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field with components $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. The **divergence** of $\mathbf{F}$ is the scalar function

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The divergence of a vector field measures how much it is "expanding" at each point.

Examples

1. If $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ then $\nabla \cdot \mathbf{F} = 2$.
2. If $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ then $\nabla \cdot \mathbf{F} = 0$.

Curl

Definition

Let $\mathbf{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field with components $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. The **curl** of $\mathbf{F}$ is the vector field

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

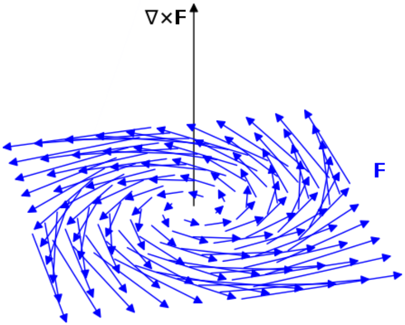
$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.$$

The magnitude of the curl, $\|\nabla \times \mathbf{F}\|$, measures how much $\mathbf{F}$ rotates around a point. The direction of the curl, $\frac{\nabla\times\mathbf{F}}{\|\nabla\times\mathbf{F}\|}$, is the axis around which it rotates.

Curl

Example

If $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ then $\nabla \times \mathbf{F} = 2\,\mathbf{k}$.

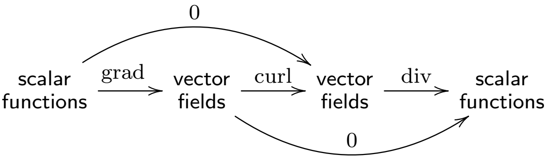


How they're related

Theorem
Let $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar function. Then $\nabla \times (\nabla f) = 0$, that is, $\operatorname{curl}(\operatorname{grad} f) = 0$.

Theorem
Let $\mathbf{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 vector field. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, that is, $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

To summarize, the composition of any two consecutive arrows in the diagram yields zero.



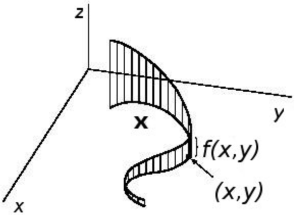
Scalar line integrals

Definition

Let $\mathbf{x} : [a, b] \to X \subseteq \mathbb{R}^3$ be a C^1 path and $f : X \to \mathbb{R}^3$ a continuous function. The **scalar line integral** of f along $\mathbf{x}$ is

$$\int_{\mathbf{x}} f\, ds = \int_a^b f(\mathbf{x}(t)) \left\| \mathbf{x}'(t) \right\| dt.$$

In two dimensions, a scalar line integral measures the area under a curve with base $\mathbf{x}$ and height given by f.



Scalar line integrals

Example

Let $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3$ be the helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ and let $f(x, y, z) = xy + z$. Let's compute

$$\int_{\mathbf{x}} f\, ds = \int_0^{2\pi} f(\mathbf{x}(t)) \left\|\mathbf{x}'(t)\right\| \, dt.$$

We find

$$\left\|\mathbf{x}'(t)\right\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2},$$

so now

$$\int_0^{2\pi} f(\mathbf{x}(t)) \left\|\mathbf{x}'(t)\right\| \, dt = \int_0^{2\pi} (\cos t \sin t + t) \sqrt{2}\, dt$$
$$= \sqrt{2} \int_0^{2\pi} \left(\tfrac{1}{2} \sin 2t + t\right) dt = 2\sqrt{2}\pi^2.$$

Vector line integrals

Definition

Let $\mathbf{x} : [a, b] \to X \subseteq \mathbb{R}^3$ be a C^1 path and $\mathbf{F} : X \to \mathbb{R}^3$ a continuous vector field. The **vector line integral** of $\mathbf{F}$ along $\mathbf{x}$ is

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

If $\mathbf{F}$ has components $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, the vector line integral can also be written

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} F_x dx + F_y dy + F_z dz.$$

Physically, a vector line integral measures the work done by the force field $\mathbf{F}$ on a particle moving along the path $\mathbf{x}$.

Vector line integrals

Example

Let $\mathbf{x} : [0,1] \to \mathbb{R}^3$ be the path $\mathbf{x}(t) = (2t+1, t, 3t-1)$ and let $\mathbf{F} = -z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$. Let's compute

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} -z dx + x dy + y dz.$$

First, we find $\mathbf{x}'(t) = (2,1,3)$, and now we can do

$$\begin{aligned}\int_{\mathbf{x}} -z dx + x dy + y dz &= \int_0^1 -(3t-1)(2) + (2t+1) + t(3) dt \\ &= \int_0^1 -t + 3 dt = \tfrac{5}{2}.\end{aligned}$$

Changing orientation

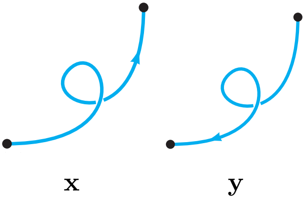


Figure: $\mathbf{x}$ and $\mathbf{y}$ have opposite orientations

$$\int_{\mathbf{y}} f\,ds = \int_{\mathbf{x}} f\,ds$$

$$\int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}$$

This can be achieved by negating t:

$$\mathbf{y}(t) = \mathbf{x}(-t).$$

Conservative vector fields

Definition

A continuous vector field $\mathbf{F}$ is called a **conservative vector field**, or a **gradient field**, if $\mathbf{F} = \nabla f$ for some C^1 scalar function f. In this case we also say that f is a **scalar potential** of $\mathbf{F}$.

Theorem

Suppose $\mathbf{F}$ *is a continuous vector field defined on a connected, open region* $R \subseteq \mathbb{R}^3$*. Then* $\mathbf{F} = \nabla f$ *if and only if* $\mathbf{F}$ *has* **path independent line integrals** *in* R*.*

Path independence

We say $\mathbf{F} : R \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ has **path independent line integrals** if any of the following hold:

1. $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$ whenever $\mathbf{x}$ and $\mathbf{y}$ are two simple C^1 paths in R with the same initial and terminal points,
2. $\oint_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = 0$ for any simple, *closed* C^1 path $\mathbf{x}$ lying in R (meaning the initial and terminal points of $\mathbf{x}$ coincide),
3. $\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A)$ for any differentiable curve C in R running from point A to point B, and for any scalar potential f.

Physical interpretation

To justify our terminology, if f is a scalar potential for the vector field $\mathbf{F}$, it means that we can interpret f as measuring the *potential* energy associated with the force represented by $\mathbf{F}$.

In this setting, criterion 3 from the previous slide says that

$$\text{work} = \int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A) = \text{change in potential energy},$$

meaning that the force represented by $\mathbf{F}$ obeys *conservation* of energy.

A test for conservative fields

Theorem

Suppose $\mathbf{F}$ *is a* C^1 *vector field defined in a* **simply-connected** *region,* R*, (intuitively,* R *has no holes going all the way through). Then* $\mathbf{F} = \nabla f$ *for some* C^2 *scalar function if and only if* $\nabla \times \mathbf{F} = \mathbf{0}$ *at all points in* R*.*

Example

Let

$$\mathbf{F} = \left(\frac{x}{x^2+y^2+z^2} - 6x\right)\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k}.$$

$\mathbf{F}$ is C^1 on $\mathbb{R}^3 - \{(0,0,0)\}$, which is a simply-connected domain. Check that

$$\nabla \times \mathbf{F} = \mathbf{0}$$

everywhere $\mathbf{F}$ is defined. Therefore, $\mathbf{F}$ is conservative.