# Row Space, Column Space, and the Rank-Nullity Theorem

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 22, 2013

# Agenda

# Motivation

Say $S$ is a subspace of $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. What operations can we perform on the basis while preserving its span and linear independence?

- Swap two elements (or shuffle them in any way)

$$\text{E.g. } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \to \{\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3\}$$

- Multiply one element by a nonzero scalar

$$\text{E.g. } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \to \{\mathbf{v}_1, 5\mathbf{v}_2, \mathbf{v}_3\}$$

- Add a scalar multiple of one element to another

$$\text{E.g. } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \to \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + 2\mathbf{v}_2\}$$

If we make the $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the rows of a matrix, these operations are just the familiar elementary row ops.

# Row space

## Definition

If $A$ is an $m \times n$ matrix with real entries, the **row space** of $A$ is the subspace of $\mathbb{R}^n$ spanned by its rows.

## Remarks

1. Elementary row ops do not change the row space.
2. In general, the rows of a matrix may not be linearly independent.

## Theorem

*The nonzero rows of any row-echelon form of $A$ is a basis for its row space.*

# Example

Determine a basis for the row space of

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 3 & -1 & 1 & 7 & 0 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}.$$

Reduce $A$ to the row-echelon form

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the row space of $A$ is the 2-dimensional subspace of $\mathbb{R}^5$ with basis

$$\{(1,-1,1,3,2),\ (0,1,-1,-1,-3)\}.$$

# Column space

We can do the same thing for columns.

## Definition

If $A$ is an $m \times n$ matrix with real entries, the **column space** of $A$ is the subspace of $\mathbb{R}^m$ spanned by its columns.

Obviously, the column space of $A$ equals the row space of $A^T$, so a basis can be computed by reducing $A^T$ to row-echelon form. However, this is not the best way.

# Example

Determine a basis for the column space of

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -1 & 1 & 0 \\ 3 & 6 & -1 & 4 & 1 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix}.$$

Reduce $A$ to the reduced row-echelon form

$$E = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \end{bmatrix}.$$

$$\mathbf{e}_2 = 2\mathbf{e}_1 \Rightarrow \mathbf{a}_2 = 2\mathbf{a}_1$$

$$\mathbf{e}_4 = 3\mathbf{e}_1 + 5\mathbf{e}_3 \Rightarrow \mathbf{a}_4 = 3\mathbf{a}_1 + 5\mathbf{a}_3$$

Therefore, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for the column space of $A$.

# Column space

We don't need to go all the way to RREF; we can see where the leading ones will be just from REF.

## Theorem

*If $A$ is an $m \times n$ matrix with real entries, the set of column vectors of $A$ corresponding to those columns containing leading ones in any row-echelon form of $A$ is a basis for the column space of $A$.*

## Another point of view

The column space of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^m$ consisting of the vectors $\mathbf{v} \in \mathbb{R}^m$ such that the linear system $A\mathbf{x} = \mathbf{v}$ is consistent.

# Relation to rank

If $A$ is an $m \times n$ matrix, to determine bases for the row space and column space of $A$, we reduce $A$ to a row-echelon form $E$.

1. The rows of $E$ containing leading ones form a basis for the row space.
2. The columns of $A$ corresponding to columns of $E$ with leading ones form a basis for the column space.

$$\dim(\text{rowspace}(A)) = \text{rank}(A) = \dim(\text{colspace}(A))$$

# Segue

If $A$ is an $m \times n$ matrix, we noted that in the linear system

$$A\mathbf{x} = \mathbf{v},$$

$\operatorname{rank}(A)$, functioning as $\dim(\operatorname{colspace}(A))$, represents the degrees of freedom in $\mathbf{v}$ while keeping the system consistent.

The degrees of freedom in $\mathbf{x}$ while keeping $\mathbf{v}$ constant is the number of free variables in the system. We know this to be $n - \operatorname{rank}(A)$, since $\operatorname{rank}(A)$ is the number of bound variables.

Freedom in choosing $\mathbf{x}$ comes from the null space of $A$, since if $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{y} = \mathbf{0}$ then

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{v} + \mathbf{0} = \mathbf{v}.$$

Hence, the degrees of freedom in $\mathbf{x}$ should be equal to $\dim(\operatorname{nullspace}(A))$.

# The Rank-Nullity Theorem

## Definition

When $A$ is an $m \times n$ matrix, recall that the null space of $A$ is

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Its dimension is referred to as the **nullity** of $A$.

## Theorem (Rank-Nullity Theorem)

*For any $m \times n$ matrix $A$,*

$$\text{rank}(A) + \text{nullity}(A) = n.$$

# Homogeneous linear systems

We're now going to examine the geometry of the solution set of a linear system. Consider the linear system

$$A\mathbf{x} = \mathbf{b},$$

where $A$ is $m \times n$.

If $\mathbf{b} = \mathbf{0}$, the system is called **homogeneous**. In this case, the solution set is simply the null space of $A$.

Any homogeneous system has the solution $\mathbf{x} = \mathbf{0}$, which is called the **trivial solution**. Geometrically, this means that the solution set passes through the origin. Furthermore, we have shown that the solution set of a homogeneous system is in fact a subspace of $\mathbb{R}^n$.

# Structure of a homogeneous solution set

## Theorem

- *If* $\operatorname{rank}(A) = n$, *then* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$, *so* $\operatorname{nullspace}(A) = \{\mathbf{0}\}$.
- *If* $\operatorname{rank}(A) = r < n$, *then* $A\mathbf{x} = \mathbf{0}$ *has an infinite number of solutions, all of which are of the form*

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r},$$

*where* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ *is a basis for* $\operatorname{nullspace}(A)$.

## Remark

Such an expression is called the **general solution** to the homogeneous linear system.

# Nonhomogeneous linear systems

Now consider a nonhomogeneous linear system

$$A\mathbf{x} = \mathbf{b}$$

where $A$ be an $m \times n$ matrix and $\mathbf{b}$ is not necessarily $\mathbf{0}$.

## Theorem

- *If $\mathbf{b}$ is not in $\operatorname{colspace}(A)$, then the system is inconsistent.*
- *If $\mathbf{b} \in \operatorname{colspace}(A)$, then the system is consistent and has*
  - *a unique solution if and only if $\operatorname{rank}(A) = n$.*
  - *an infinite number of solutions if and only if $\operatorname{rank}(A) < n$.*

Geometrically, a nonhomogeneous solution set is just the corresponding homogeneous solution set that has been shifted away from the origin.

# Structure of a nonhomogeneous solution set

## Theorem

*In the case where* $\operatorname{rank}(A) = r < n$ *and* $\mathbf{b} \in \operatorname{colspace}(A)$*, then all solutions are of the form*

$$\begin{aligned} \mathbf{x} &= \underbrace{c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r}}_{\mathbf{x}_c} + \mathbf{x}_p, \\ &= \qquad\qquad \mathbf{x}_c \qquad\qquad + \mathbf{x}_p \end{aligned}$$

*where* $\mathbf{x}_p$ *is any particular solution to* $A\mathbf{x} = \mathbf{b}$ *and* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ *is a basis for* $\operatorname{nullspace}(A)$*.*

## Remark

The above expression is the **general solution** to a nonhomogeneous linear system. It has two components:

- the **complementary solution**, $\mathbf{x}_c$, and
- the **particular solution**, $\mathbf{x}_p$.