Vector Differential Equations: Defective Coefficient Matrix and Higher Order Linear Differential Equations

Math 240 — Calculus III

Summer 2013, Session II

Thursday, August 1, 2013

Agenda

1. Vector differential equations: defective coefficient matrix

2. Linear differential equations of order n

   Linear differential operators

   Familiar stuff

   A taste of what's to come

Introduction

We've learned how to find a matrix S so that $S^{-1}AS$ is almost a diagonal matrix. Recall that diagonalization allows us to solve linear systems of diff. eqs. because we can solve the equation

$$y' = ay.$$

Jordan form will give us small systems that look like

$$\begin{aligned} y_1' &= ay_1 + y_2, \\ y_2' &= \qquad\quad ay_2. \end{aligned}$$

Is there an obvious solution?

$$y_1(t) = e^{at} \text{ and } y_2(t) = 0.$$

A nontrivial one? Yes!

$$y_1(t) = te^{at} \text{ and } y_2(t) = e^{at}.$$

Write this in the vector form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{at} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

2×2 defective systems

Switching back to the standard basis, these are the solutions

$$\mathbf{x}_1(t) = e^{at}\mathbf{v}_1 \text{ and } \mathbf{x}_2(t) = e^{at}(t\mathbf{v}_1 + \mathbf{v}_2)$$

where $\mathbf{v}_2, \mathbf{v}_1$ is a chain of generalized eigenvectors.

Example

Find the general solution to

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}.$$

1. The single eigenvalue is $\lambda = 3$.
2. Chain of generalized e-vectors is $\mathbf{v}_1 = (1, 3)$, $\mathbf{v}_2 = (0, 1)$.
$$(A - 3I)\mathbf{v}_1 = \mathbf{0} \text{ and } (A - 3I)\mathbf{v}_2 = \mathbf{v}_1.$$
3. Fundamental set of solutions is therefore
$$\mathbf{x}_1(t) = e^{3t}\mathbf{v}_1 \text{ and } \mathbf{x}_2(t) = e^{3t}\left(t\mathbf{v}_1 + \mathbf{v}_2\right).$$

Longer chains

What about chains of generalized eigenvectors longer than 2?

If A is an $n \times n$ matrix with eigenvalue λ and chain of generalized eigenvectors

$$\mathbf{v}_1 = (A - \lambda I)^{p-1}\mathbf{v}, \qquad \mathbf{v}_2 = (A - \lambda I)^{p-2}\mathbf{v}, \quad \ldots$$
$$\mathbf{v}_{p-1} = (A - \lambda I)\mathbf{v}, \qquad \mathbf{v}_p = \mathbf{v},$$

check that the following are solutions to $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$$
$$\mathbf{x}_2(t) = e^{\lambda t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right)$$
$$\vdots$$
$$\mathbf{x}_p(t) = e^{\lambda t}\left(\mathbf{v}_p + t\mathbf{v}_{p-1} + \cdots + \frac{1}{(p-1)!}t^{p-1}\mathbf{v}_1\right)$$

Longer chains

We should also check that $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_p(t)\}$ is independent. We know that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is independent, that is,

$$\det\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}\right) \neq 0.$$

Theorem

The set $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_p(t)\}$ is a linearly independent subset of $V_n(I)$.

Thus, we can construct a fundamental set of solutions by applying the foregoing construction to each chain of generalized eigenvectors.

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. Only eigenvalue is $\lambda = 1$.
2. Yesterday we found the chain

$$\mathbf{v}_1 = (-2, 0, 1),\ \mathbf{v}_2 = (0, -1, 0),\ \mathbf{v}_3 = (-1, 0, 0).$$

3. Thus, solutions are

$$\begin{aligned} \mathbf{x}_1(t) &= e^t \mathbf{v}_1, \\ \mathbf{x}_2(t) &= e^t \left(\mathbf{v}_2 + t\mathbf{v}_1\right), \\ \mathbf{x}_3(t) &= e^t \left(\mathbf{v}_3 + t\mathbf{v}_2 + \tfrac{1}{2}t^2\mathbf{v}_3\right). \end{aligned}$$

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

1. Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$.
2. Eigenvectors and generalized eigenvectors are

$$A\mathbf{e}_1 = 2\mathbf{e}_1, \quad A\mathbf{e}_2 = 2\mathbf{e}_2 + \mathbf{e}_1, \quad A\mathbf{e}_3 = 5\mathbf{e}_3,$$
$$A\mathbf{e}_4 = 5\mathbf{e}_4, \quad A\mathbf{e}_5 = 5\mathbf{e}_5 + \mathbf{e}_4, \quad A\mathbf{e}_6 = 5\mathbf{e}_6 + \mathbf{e}_5.$$

3. Our fundamental set of solutions is

$$\mathbf{x}_1(t) = e^{2t}\mathbf{e}_1, \quad \mathbf{x}_2(t) = e^{2t}\left(\mathbf{e}_2 + t\mathbf{e}_1\right), \quad \mathbf{x}_3(t) = e^{5t}\mathbf{e}_3,$$
$$\mathbf{x}_4(t) = e^{5t}\mathbf{e}_4, \quad \mathbf{x}_5(t) = e^{5t}\left(\mathbf{e}_5 + t\mathbf{e}_4\right),$$
$$\mathbf{x}_6(t) = e^{5t}\left(\mathbf{e}_6 + t\mathbf{e}_5 + \tfrac{1}{2}t^2\mathbf{e}_6\right).$$

Introduction

We now turn our attention to solving **linear differential equations of order** n. The general form of such an equation is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n,$ and F are functions defined on an interval I.

The general strategy is to reformulate the above equation as

$$Ly = F,$$

where L is an appropriate linear transformation. In fact, L will be a *linear differential operator*.

Linear differential operators

Recall that the mapping $D : C^1(I) \to C^0(I)$ defined by $D(f) = f'$ is a linear transformation. This D is called the **derivative operator.** Higher order derivative operators $D^k : C^k(I) \to C^0(I)$ are defined by composition:

$$D^k = D \circ D^{k-1},$$

so that

$$D^k(f) = \frac{d^k f}{dx^k}.$$

A **linear differential operator of order** n is a linear combination of derivative operators of order up to n,

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n,$$

defined by

$$Ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y,$$

where the a_i are continous functions of x. L is then a linear transformation $L : C^n(I) \to C^0(I)$. (Why?)

Examples

Example

If $L = D^2 + 4xD - 3x$, then

$$Ly = y'' + 4xy' - 3xy.$$

We have

$$\begin{aligned} L(\sin x) &= -\sin x + 4x\cos x - 3x\sin x, \\ L\left(x^2\right) &= 2 + 8x^2 - 3x^3. \end{aligned}$$

Example

If $L = D^2 - e^{3x}D$, determine

1. $L\left(2x - 3e^{2x}\right) = -12e^{2x} - 2e^{3x} + 6e^{5x}$
2. $L\left(3\sin^2 x\right) = -3e^{3x}\sin 2x - 6\cos 2x$

Homogeneous and nonhomogeneous equations

Consider the general n-th order linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0 \neq 0$ and $a_0, a_1, \ldots, a_n,$ and F are functions on an interval I.

If $a_0(x)$ is nonzero on I, then we may divide by it and relabel, obtaining

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

which we rewrite as

$$Ly = F(x),$$

where $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$.

If $F(x)$ is identically zero on I, then the equation is **homogeneous**, otherwise it is **nonhomogeneous**.

The general solution

If we have a homogeneous linear differential equation

$$Ly = 0,$$

its solution set will coincide with $\mathrm{Ker}(L)$. In particular, the kernel of a linear transformation is a subspace of its domain.

Theorem

The set of solutions to a linear differential equation of order n is a subspace of $C^n(I)$. It is called the **solution space**. *The dimension of the solutions space is n.*

Being a vector space, the solution space has a basis $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ consisting of n solutions. Any element of the vector space can be written as a linear combination of basis vectors

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

This expression is called the **general solution**.

The Wronskian

We can use the Wronskian

$$W[y_1, y_2, \ldots, y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

to determine whether a set of solutions is linearly independent.

Theorem

Let $y_1, y_2, \ldots, y_n$ be solutions to the n-th order differential equation $Ly = 0$ whose coefficients are continuous on I. If $W[y_1, y_2, \ldots, y_n](x) = 0$ at any single point $x \in I$, then $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent.

To summarize, the vanishing or nonvanishing of the Wronskian on an interval *completely characterizes* the linear dependence or independence of a set of solutions to $Ly = 0$.

The Wronskian

Example

Verify that $y_1(x) = \cos 2x$ and $y_2(x) = 3(1 - 2\sin^2 x)$ are solutions to the differential equation $y'' + 4y = 0$ on $(-\infty, \infty)$.

Determine whether they are linearly independent on this interval.

$$W[y_1, y_2](x) = \begin{vmatrix} \cos 2x & 3(1 - 2\sin^2 x) \\ -2\sin 2x & -12\sin x \cos x \end{vmatrix}$$

$$= -6\sin 2x \cos 2x + 6\sin 2x \cos 2x = 0$$

They are linearly dependent. In fact, $3y_1 - y_2 = 0$.

Nonhomogeneous equations

Consider the nonhomogeneous linear differential equation $Ly = F$. The **associated homogeneous equation** is $Ly = 0$.

Theorem

Suppose $\{y_1, y_2, \ldots, y_n\}$ are n linearly independent solutions to the n-th order equation $Ly = 0$ on an interval I, and $y = y_p$ is any particular solution to $Ly = F$ on I. Then every solution to $Ly = F$ on I is of the form

$$\begin{aligned} y &= \underbrace{c_1y_1 + c_2y_2 + \cdots + c_ny_n}_{y_c} + y_p, \\ &= \qquad\qquad y_c \qquad\qquad + y_p \end{aligned}$$

for appropriate constants $c_1, c_2, \ldots, c_n$.

This expression is the **general solution** to $Ly = F$. The components of the general solution are

- the **complementary function**, y_c, which is the general solution to the associated homogeneous equation,
- the **particular solution**, y_p.

Something slightly new

Theorem

If $y = u_p$ *and* $y = v_p$ *are particular solutions to* $Ly = f(x)$ *and* $Ly = g(x)$*, respectively, then* $y = u_p + v_p$ *is a solution to* $Ly = f(x) + g(x)$*.*

Proof.

We have $L(u_p + v_p) = L(u_p) + L(v_p) = f(x) + g(x)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

A taste of what's to come

Example

Determine all solutions to the differential equation $y'' + y' - 6y = 0$ of the form $y(x) = e^{rx}$, where r is a constant.

Substituting $y(x) = e^{rx}$ into the equation yields

$$e^{rx}(r^2 + r - 6) = r^2 e^{rx} + re^{rx} - 6e^{rx} = 0.$$

Since $e^{rx} \neq 0$, we just need $(r+3)(r-2) = 0$. Hence, the two solutions of this form are

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-3x}.$$

Could this be a basis for the solution space? Check linear independence. Yes! The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}.$$

A taste of what's to come

Example

Determine the general solution to the differential equation

$$y'' + y' - 6y = 8e^{5x}.$$

We know the complementary function,

$$y_c(x) = c_1e^{2x} + c_2e^{-3x}.$$

For the particular solution, we might guess something of the form $y_p(x) = ce^{5x}$. What should c be? We want

$$8e^{5x} = y_p'' + y_p' - 6y_p = (25c + 5c - 6c)e^{5x}.$$

Cancel e^{5x} and then solve $8 = 24c$ to find $c = \frac{1}{3}$.

The general solution is

$$y(x) = c_1e^{2x} + c_2e^{-3x} + \tfrac{1}{3}e^{5x}.$$