Linear Algebra Practice Problems

Math 240 — Calculus III

Summer 2015, Session II

1. Determine whether the given set is a vector space. If not, give at least one axiom that is not satisfied. Unless otherwise stated, assume that vector addition and scalar multiplication are the ordinary operations defined on the set.

   (a) The set of vectors $\{(a, b) \in \mathbb{R}^2 : b = 3a + 1\}$
   Answer: This is not a vector space. It does not contain the zero vector, and is not closed under either addition or scalar multiplication.

   (b) The set of vectors $\{(a, b) \in \mathbb{R}^2\}$ with scalar multiplication defined by $k(a, b) = (ka, b)$
   Answer: This is not a vector space. The scalar multiplication defined above does not distribute over the usual addition of vectors.

   $$(r + s)(a, b) = ((r + s)a, b) = (ra + sa, b)$$
   $$\text{but } r(a, b) + s(a, b) = (ra, b) + (sa, b) = (ra + sa, 2b)$$

   (c) The set of vectors $\{(a, b) \in \mathbb{R}^2\}$ with scalar multiplication defined by $k(a, b) = (ka, 0)$
   Answer: This is not a vector space. It does not obey the identity property of scalar multiplication because
   $$1(a, b) = (1a, 0) = (a, 0) \neq (a, b).$$

   (d) The set of real numbers, with addition defined by $\mathbf{x} + \mathbf{y} = x - y$
   Answer: This is not a vector space. This method of vector addition is neither associative nor commutative.

   (e) The set $\mathbb{R}^3$, with the vector addition operation $\oplus$ defined by

   $$(a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1 + 5, a_2 + b_2 - 7, a_3 + b_3 + 1)$$

   and scalar multiplication $\odot$ defined by

   $$c \odot (a_1, a_2, a_3) = (ca_1 + 5(c - 1), ca_2 - 7(c - 1), ca_3 + c - 1).$$

   Answer: This is a vector space. The zero vector is $(-5, 7, -1)$.

2. Determine whether or not the given set is a subspace of the indicated vector space.

   (a) $\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$
   Answer: This is not a subspace of $\mathbb{R}^3$. It does not contain the zero vector $\mathbf{0} = (0, 0, 0)$ and it is not closed under either addition or scalar multiplication.

   (b) All polynomials in P_2 that are divisible by $x - 2$
   Answer: This is a subspace of P_2.

(c) $\{f \in C^0[a,b] : \int_a^b f(x)\,dx = 0\}$

Remember that $C^0[a,b]$ is the vector space of continuous, real-valued functions defined on the closed interval $[a,b]$ with $a < b$.

Answer: This is a subspace of $C^0[a,b]$. It is the kernel of the linear transformation $T : C^0[a,b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\,dx$.

3. If $A = \begin{bmatrix} 1 & 4 \\ 5 & 10 \\ 8 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 6 & -3 \\ 1 & -3 & 2 \end{bmatrix}$, determine (a) AB and (b) BA.

Answer: (a) $AB = \begin{bmatrix} 0 & -6 & 5 \\ -10 & 0 & 5 \\ -20 & 12 & 0 \end{bmatrix}$, (b) $BA = \begin{bmatrix} 2 & 8 \\ 2 & -2 \end{bmatrix}$

4. Use either Gaussian elimination or Gauss-Jordan elimination to solve the given system or show that no solution exists.

(a)

$$\begin{aligned} x_1 - x_2 - x_3 &= -3 \\ 2x_1 + 3x_2 + 5x_3 &= 7 \\ x_1 - 2x_2 + 3x_3 &= -11 \end{aligned}$$

Answer: $x_1 = 0,\ x_2 = 4,\ x_3 = -1$

(b)

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Answer: $x_1 = t,\ x_2 = -t,\ x_3 = 0$ for any $t \in \mathbb{R}$

(c)

$$\begin{aligned} x_1 - x_2 - x_3 &= 8 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Answer: The system is inconsistent; there is no solution.

(d)

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+ x_3} + x_4 &= 0 \\ 4x_1 + 9x_2 + x_3 + 12x_4 &= 0 \\ 3x_1 + 9x_2 + 6x_3 + 21x_4 &= 0 \\ x_1 + 3x_2 + x_3 + 9x_4 &= 0 \end{aligned}$$

Answer: $x_1 = 19t,\ x_2 = -10t,\ x_3 = 2t,\ x_4 = t$ for any $t \in \mathbb{R}$

5. Determine the rank of the given matrix.

(a) $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Answer: 2

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix}$

Answer: 3

(c) $\begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{bmatrix}$

Answer: 3

6. Determine whether the given set of vectors in $\mathbb{R}^n$ is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (1,-1,5)$

Answer: These vectors are linearly independent.

(b) $\mathbf{v}_1 = (2,6,3)$, $\mathbf{v}_2 = (1,-1,4)$, $\mathbf{v}_3 = (3,2,1)$, $\mathbf{v}_4 = (2,5,4)$

Answer: These vectors are linearly dependent. They are vectors in $\mathbb{R}^3$, which is a 3-dimensional vector space. Any set of more than 3 vectors in $\mathbb{R}^3$ is linearly dependent.

(c) $\mathbf{v}_1 = (1,-1,3,-1)$, $\mathbf{v}_2 = (1,-1,4,2)$, $\mathbf{v}_3 = (1,-1,5,7)$.

Answer: These vectors are linearly independent.

(d) $\mathbf{v}_1 = (2, 1, 1, 5)$, $\mathbf{v}_2 = (2, 2, 1, 1)$, $\mathbf{v}_3 = (3, -1, 6, 1)$, $\mathbf{v}_4 = (1, 1, 1, -1)$
Answer: These vectors are linearly independent.

7. Determine a basis for the subspace of $\mathbb{R}^n$ spanned by the given set of vectors.

(a) $\{(1, 3, 3),\ (-3, -9, -9),\ (1, 5, -1),\ (2, 7, 4),\ (1, 4, 1)\}$
Answer: $\{(1, 3, 3),\ (1, 5, -1)\}$ (answers may vary)

(b) $\{(1, 1, -1, 2),\ (2, 1, 3, -4),\ (1, 2, -6, 10)\}$
Answer: $\{(1, 1, -1, 2),\ (2, 1, 3, -4)\}$ (answers may vary)

8. Evaluate the determinant of the given matrix.

(a) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{bmatrix}$
Answer: -48

(b) $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{bmatrix}$
Answer: 62

(c) $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$
Answer: 0

(d) $\begin{bmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{bmatrix}$
Answer: -85

(e) $\begin{bmatrix} 6 & 1 & 8 & 10 \\ 0 & \frac{2}{3} & 7 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -5 \end{bmatrix}$
Answer: 80

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{bmatrix}$
Answer: 16

9. Find the values of λ that satisfy the equation

$$\begin{vmatrix} -3-\lambda & 10 \\ 2 & 5-\lambda \end{vmatrix} = 0.$$

Answer: $\lambda = -5, 7$

10. If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5$, evaluate the determinant of the matrix

$$\begin{bmatrix} 2a_1 & a_2 & a_3 \\ 6b_1 & 3b_2 & 3b_3 \\ 2c_1 & c_2 & c_3 \end{bmatrix}.$$

Answer: 30

11. Recall that a square matrix A is said to be **skew-symmetric** if $A^T = -A$. If A is a 5×5 skew-symmetric matrix, show that $\det(A) = 0$.

Answer: We know that $\det\left(A^T\right) = \det(A)$. Also, if A is 5×5, then $\det(-A) = (-1)^5 \det(A)$. Putting these together with the information $A^T = -A$, we get

$$\det(A) = \det\left(A^T\right) = \det(-A) = (-1)^5 \det(A) = -\det(A).$$

The only number that is equal to its negative is 0.

12. Determine whether the given matrix is singular or nonsingular. If it is nonsingular, find its inverse.

(a) $\begin{bmatrix} 6 & 0 \\ -3 & 2 \end{bmatrix}$

Answer: $\begin{bmatrix} 1/6 & 0 \\ 1/4 & 1/2 \end{bmatrix}$

(b) $\begin{bmatrix} -2\pi & -\pi \\ -\pi & \pi \end{bmatrix}$

Answer: $\dfrac{-1}{3\pi}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 5 & 1 \end{bmatrix}$

Answer:
$\begin{bmatrix} 7/15 & -13/30 & -8/15 \\ 1/15 & -2/15 & 1/15 \\ 2/15 & 7/30 & 2/15 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Answer:
$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$

Answer:
$\dfrac{1}{3}\begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -1 & 1 \\ -1 & 5 & 0 \\ 0 & 6 & -1 \end{bmatrix}$

Answer: This matrix is singular.

13. Use an inverse matrix to solve the linear system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 2x_2 + 2x_3 &= -3 \\ 3x_1 - x_2 + 5x_3 &= 7 \end{aligned}$$

Answer: $x_1 = 21,\ x_2 = 1,\ x_3 = -11$

14. Write the linear system

$$\begin{aligned} 7x_1 - 2x_2 &= b_1, \\ 3x_1 - 2x_2 &= b_2, \end{aligned}$$

in the form $A\mathbf{x} = \mathbf{b}$. Use $\mathbf{x} = A^{-1}\mathbf{b}$ to solve the system for each $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 10 \\ 50 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}.$$

Answer: $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 7 & -2 \\ 3 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ 3/8 & -7/8 \end{bmatrix},\quad A^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -13/8 \end{bmatrix},\quad A^{-1}\begin{bmatrix} 10 \\ 50 \end{bmatrix} = \begin{bmatrix} -10 \\ -40 \end{bmatrix},\quad A^{-1}\begin{bmatrix} 0 \\ -20 \end{bmatrix} = \begin{bmatrix} 5 \\ 35/2 \end{bmatrix}$$

15. Determine the matrix representation for the given linear transformation T relative to the ordered bases B and C.

(a) $T : M_2(\mathbb{R}) \to P_3$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-d) + 3bx^2 + (c-a)x^3$ with

i. $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and $C = \{1, x, x^2, x^3\}$,

Answer: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

ii. $B = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ and $C = \{x, 1, x^3, x^2\}$.

Answer: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(b) $T : V \to V$, where $V = \mathrm{span}\{e^{2x}, e^{-3x}\}$, given by $T(f) = f'$ with

i. $B = C = \{e^{2x}, e^{-3x}\}$,

Answer: $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

ii. $B = \{e^{2x} - 3e^{-3x}, 2e^{-3x}\}$ and $C = \{e^{2x} + e^{-3x}, -e^{2x}\}$.

Answer: $\begin{bmatrix} 9 & -6 \\ 7 & -6 \end{bmatrix}$

16. Determine which of the indicated column vectors are eigenvectors of the given matrix A. Give the corresponding eigenvalue for each one that is.

(a) $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$, $\mathbf{v}_1 = (5, -2),\ \mathbf{v}_2 = (2, 5),\ \mathbf{v}_3 = (-2, 5)$

Answer: $\mathbf{v}_3$ is an eigenvector for the eigenvalue $\lambda = -1$.

(b) $A = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$, $\mathbf{v}_1 = (1, 2 - \sqrt{2}),\ \mathbf{v}_2 = (2 + \sqrt{2}, 2),\ \mathbf{v}_3 = (\sqrt{2}, -\sqrt{2})$

Answer: $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalue $\lambda = \sqrt{2}$.

(c) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$, $\mathbf{v}_1 = (0, 0),\ \mathbf{v}_2 = (2 + 2i, -1),\ \mathbf{v}_3 = (2 + 2i, 1)$

Answer: $\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda = 2i$.

Note: The zero vector is not allowed as an eigenvector.

(d) $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$, $\mathbf{v}_1 = (-1, 4, 3),\ \mathbf{v}_2 = (1, 4, 3),\ \mathbf{v}_3 = (3, 1, 4)$

Answer: $\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda = 3$.

17. Find the eigenvalues and eigenvectors of the given matrix.

(a) $\begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix}$

Answer: $\lambda_1 = 1,\ \mathbf{v}_1 = (1,1),\ \lambda_2 = 6,\ \mathbf{v}_2 = (2,7)$

(b) $\begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix}$

Answer: $\lambda_1 = 3i,\ \mathbf{v}_1 = (2, 1+3i),\ \lambda_2 = -3i,\ \mathbf{v}_2 = (2, 1-3i)$

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Answer: $\lambda_1 = 1+i,\ \mathbf{v}_1 = (i,1),\ \lambda_2 = 1-i,\ \mathbf{v}_2 = (-i,1)$

(d) $\begin{bmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{bmatrix}$

Answer: $\lambda_1 = 4,\ \mathbf{v}_1 = (1,1,1),\ \lambda_2 = -4,\ \mathbf{v}_2 = (1,9,1),\ \lambda_3 = 0,\ \mathbf{v}_3 = (9,45,25)$

(e) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

Answer: $\lambda_1 = -1,\ \mathbf{v}_1 = (1,-1,1),\ \lambda_2 = i,\ \mathbf{v}_2 = (i,1,1),\ \lambda_3 = -i,\ \mathbf{v}_3 = (-i,1,1)$

(f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{bmatrix}$

Answer: $\lambda_1 = 1,\ \mathbf{v}_1 = (1,0,0),\ \lambda_2 = 5,\ \mathbf{v}_2 = (1,2,0),\ \lambda_3 = -7,\ \mathbf{v}_3 = (1,2,-4)$

18. Determine whether the given matrix A is diagonalizable. If so, find the matrix P that diagonalizes A and the diagonal matrix D such that $D = P^{-1}AP$.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

Answer: This matrix has one eigenvalue with algebraic multiplicity 2, but only one linearly independent eigenvector. It is defective and therefore not diagonalizable.

(b) $\begin{bmatrix} -9 & 13 \\ -2 & 6 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 13 \\ 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 4 & 0 \\ 0 & -7 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},\ D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

(d) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(e) $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: This matrix has two eigenvalues with algebraic multiplicities of 1 and 2, respectively. The latter has only one linearly independent eigenvector, hence the matrix is defective and not diagonalizable.

(f) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & -\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

19. Use diagonalization to solve the given system of differential equations.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} 3c^1 e^{7t} + 2c_2 e^{-4t} \\ c_1 e^{7t} - 3c_2 e^{-4t} \end{bmatrix}$$

(b) $\mathbf{x}' = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-4t} \\ c_1 e^{2t} - c_2 e^{-4t} \\ c_1 e^{2t} + c_3 e^{6t} \end{bmatrix}$$

(c) $\mathbf{x}' = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} c_1 e^{2\sqrt{2}t} + c_2 e^{-2\sqrt{2}t} + c_3 \\ \sqrt{2} c_1 e^{2\sqrt{2}t} - \sqrt{2} c_2 e^{-2\sqrt{2}t} \\ c_1 e^{2\sqrt{2}t} + c_2 e^{-2\sqrt{2}t} - c_3 \end{bmatrix}$$