Span, Linear Independence, and Dimension

Math 240 — Calculus III

Summer 2015, Session II

Monday, July 13, 2015

Agenda

1. Spanning sets
2. Linear independence
3. Bases and Dimension

Span

Last time, we described subspaces as $\{\mathbf{v} \in V : \text{stuff}\}$. Here's another way to construct subspaces:

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a set of vectors in a vector space V. A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, \ldots, c_n$ are scalars. The **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the set of all linear combinations of them.

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in V : c_1, \ldots, c_n \in \mathbb{R}\}$$

Example

The span of a single, nonzero vector is a line through the origin.

$$\operatorname{span}\{\mathbf{v}\} = \{t\mathbf{v} \in V : t \in \mathbb{R}\}$$

Span

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a subspace of V.

Question

What's the span of $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (2, -1)$ in $\mathbb{R}^2$?

Answer: $\mathbb{R}^2$.

Bigger Question

When is $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ equal to the whole vector space?

Definition

Definition

Let V be a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a **spanning set** for V if

$$\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = V.$$

We also say that V is **generated** or **spanned** by $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Theorem

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans $\mathbb{R}^n$ if and only if, for the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$, the linear system $A\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v} \in \mathbb{R}^n$.

Example

Determine whether the vectors $\mathbf{v}_1 = (1, -1, 4)$, $\mathbf{v}_2 = (-2, 1, 3)$, and $\mathbf{v}_3 = (4, -3, 5)$ span $\mathbb{R}^3$.

Our aim is to solve the linear system $A\mathbf{x} = \mathbf{v}$, where

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

for an arbitrary $\mathbf{v} \in \mathbb{R}^3$. If $\mathbf{v} = (x, y, z)$, reduce the augmented matrix to

$$\begin{bmatrix} 1 & -2 & 4 & x \\ 0 & 1 & -1 & -x - y \\ 0 & 0 & 0 & 7x + 11y + z \end{bmatrix}.$$

This has a solution only when $7x + 11y + z = 0$. Thus, the span of these three vectors is a plane; they do not span $\mathbb{R}^3$.

Linear dependence

Observe that $\{(1,0),(0,1)\}$ and $\{(1,0),(0,1),(1,2)\}$ are both spanning sets for $\mathbb{R}^2$. The latter has an "extra" vector: $(1,2)$ which is unnecessary to span $\mathbb{R}^2$. This can be seen from the relation

$$(1,2) = 1(1,0) + 2(0,1).$$

Theorem

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of at least two vectors in a vector space V. If one of the vectors in the set is a linear combination of the others, then that vector can be deleted from the set without diminishing its span.

The condition of one vector being a linear combinations of the others is called **linear dependence**.

Definition

Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is said to be **linearly dependent** if there are scalars $c_1, \ldots, c_n$, *not all zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Such a linear combination is called a **linear dependence relation** or a **linear dependency**. The set of vectors is **linearly independent** if the *only* linear combination producing $\mathbf{0}$ is the trivial one with $c_1 = \cdots = c_n = 0$.

Example

Consider a set consisting of a single vector $\mathbf{v}$.

- If $\mathbf{v} = \mathbf{0}$ then $\{\mathbf{v}\}$ is linearly dependent because, for example, $1\mathbf{v} = \mathbf{0}$.
- If $\mathbf{v} \neq \mathbf{0}$ then the only scalar c such that $c\mathbf{v} = \mathbf{0}$ is $c = 0$. Hence, $\{\mathbf{v}\}$ is linearly independent.

The zero vector and linear dependence

Theorem

A set consisting of a single vector $\mathbf{v}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$. Therefore, any set consisting of a single nonzero vector is linearly independent.

In fact, including $\mathbf{0}$ in any set of vectors will produce the linear dependency

$$\mathbf{0} + 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Theorem

Any set of vectors that includes the zero vector is linearly dependent.

Practice

1. Find a linear dependency among the vectors
$$f_1(x) = 1, \quad f_2(x) = 2\sin^2 x, \quad f_3(x) = -5\cos^2 x$$
in the vector space $C^0(\mathbb{R})$.
2. If $\mathbf{v}_1 = (1, 2, -1)$, $\mathbf{v}_2 = (2, -1, 1)$, and $\mathbf{v}_3 = (8, 1, 1)$, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in $\mathbb{R}^3$ by exhibiting a linear dependency.

Proposition

Any set of vectors contains a linearly independent subset with the same span.

Proof.

Remove $\mathbf{0}$ and any vectors that are linear combinations of the others. $\mathcal{Q.E.D.}$

Criteria for linear dependence

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if the linear system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Corollary

1. *If $k > n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.*
2. *If $k = n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if $\det(A) = 0$.*

Linear independence of functions

Definition

A set of functions $\{f_1, f_2, \ldots, f_n\}$ is **linearly independent on an interval** I if the only values of the scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \text{ for all } x \in I$$

are $c_1 = c_2 = \cdots = c_n = 0$.

Definition

Let $f_1, f_2, \ldots, f_n \in C^{n-1}(I)$. The **Wronskian** of these functions is

$$W[f_1, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Linear independence of functions

Theorem

Let $f_1, f_2, \ldots, f_n \in C^{n-1}(I)$*. If* $W[f_1, f_2, \ldots, f_n]$ *is nonzero at some point in* I *then* $\{f_1, \ldots, f_n\}$ *is linearly independent on* I*.*

Remarks

1. In order for $\{f_1, \ldots, f_n\}$ to be linearly independent on I, it is enough for $W[f_1, \ldots, f_n]$ to be nonzero at a single point.
2. The theorem *does not say* that the set is linearly dependent if $W[f_1, \ldots, f_n](x) = 0$ for all $x \in I$.
3. The Wronskian will be more useful in the case where $f_1, \ldots, f_n$ are the solutions to a differential equation, in which case it will completely determine their linear dependence or independence.

Minimal spanning sets

Since we can remove vectors from a linearly dependent set without changing the span, a "minimal spanning set" should be linearly independent.

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a **basis** (plural **bases**) for V if

1. The vectors are linearly independent.
2. They span V.

Examples

1. The **standard basis** for $\mathbb{R}^n$ is

$$\mathbf{e}_1 = (1, 0, 0, \ldots), \ \mathbf{e}_2 = (0, 1, 0, \ldots), \ \ldots$$

2. Any linearly independent set is a basis for its span.

Practice

1. Find a basis for $M_2(\mathbb{R})$.
2. Find a basis for P_2.
   In general, the standard basis for P_n is

$$\{1,\ x,\ x^2,\ \ldots,\ x^n\}.$$

Dimension

$\mathbb{R}^3$ has a basis with 3 vectors. Could any basis have more? Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is another basis for $\mathbb{R}^3$ and $n > 3$. Express each $\mathbf{v}_j$ as

$$\mathbf{v}_i = (v_{1j}, v_{2j}, v_{3j}) = v_{1j}\mathbf{e}_1 + v_{2j}\mathbf{e}_2 + v_{3j}\mathbf{e}_3.$$

If

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = [v_{ij}]$$

then the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution because $\operatorname{rank}(A) \leq 3$. Such a nontrivial solution is a linear dependency among $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so in fact they do not form a basis.

Theorem

If a vector space has a basis consisting of m vectors, then any set of more than m vectors is linearly dependent.

Dimension

Corollary

Any two bases for a single vector space contain the same number of vectors.

Definition

The number of vectors in any chosen basis is the **dimension** of the vector space. We denote it $\dim V$.

Examples

1. $\dim \mathbb{R}^n = n$
2. $\dim M_{m\times n}(\mathbb{R}) = mn$
3. $\dim P_n = n+1$
4. $\dim P = \infty$
5. $\dim C^k(I) = \infty$
6. $\dim\{\mathbf{0}\} = 0$

A vector space is called **finite dimensional** if it has a basis with a finite number of elements, or **infinite dimensional** otherwise.

Dimension

Theorem

If $\dim V = n$, *then any set of* n *linearly independent vectors in* V *is a basis.*

Theorem

If $\dim V = n$, *then any set of* n *vectors that spans* V *is a basis.*

Corollary

If S *is a subspace of a vector space* V *then*

$$\dim S \leq \dim V$$

and $S = V$ *only if* $\dim S = \dim V$.