Generalized Eigenvectors

Math 240 — Calculus III

Summer 2015, Session II

Thursday, July 23, 2015

Agenda

Motivation

Defective matrices cannot be diagonalized because they do not possess enough eigenvectors to make a basis. How can we correct this defect?

Example

The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is defective.

1. Only eigenvalue is $\lambda = 1$.
2. $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
3. Single eigenvector $\mathbf{v} = (1, 0)$.
4. We could use $\mathbf{u} = (0, 1)$ to complete a basis.
5. Notice that $(A - I)\mathbf{u} = \mathbf{v}$ and $(A - I)^2\mathbf{u} = \mathbf{0}$.

Maybe we just didn't multiply by $A - \lambda I$ enough times.

Definition

Definition

If A is an $n \times n$ matrix, a **generalized eigenvector** of A corresponding to the eigenvalue λ is a nonzero vector $\mathbf{x}$ satisfying

$$(A - \lambda I)^p \, \mathbf{x} = \mathbf{0}$$

for some positive integer p. Equivalently, it is a nonzero element of the nullspace of $(A - \lambda I)^p$.

Example

- Eigenvectors are generalized eigenvectors with $p = 1$.
- In the previous example we saw that $\mathbf{v} = (1, 0)$ and $\mathbf{u} = (0, 1)$ are generalized eigenvectors for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \lambda = 1.$$

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

1. Characteristic polynomial is $(3-\lambda)(1-\lambda)^2$.
2. Eigenvalues are $\lambda = 1, 3$.
3. Eigenvectors are

$$\begin{aligned} \lambda_1 = 3 : \qquad & \mathbf{v}_1 = (1,2,2), \\ \lambda_2 = 1 : \qquad & \mathbf{v}_2 = (1,0,0). \end{aligned}$$

4. Final generalized eigenvector will a vector $\mathbf{v}_3 \neq \mathbf{0}$ such that

$$(A - \lambda_2 I)^2 \mathbf{v}_3 = \mathbf{0} \text{ but } (A - \lambda_2 I)\mathbf{v}_3 \neq \mathbf{0}.$$

Pick $\mathbf{v}_3 = (0,1,0)$. Note that $(A - \lambda_2 I)\mathbf{v}_3 = \mathbf{v}_2$.

Facts about generalized eigenvectors

How many powers of $(A - \lambda I)$ do we need to compute in order to find all of the generalized eigenvectors for λ?

Fact

If A is an $n \times n$ matrix and λ is an eigenvalue with algebraic multiplicity k, then the set of generalized eigenvectors for λ consists of the nonzero elements of $\operatorname{nullspace}\left((A - \lambda I)^k\right)$.

In other words, we need to take at most k powers of $A - \lambda I$ to find all of the generalized eigenvectors for λ.

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. Single eigenvalue of $\lambda = 1$.
2. Single eigenvector $\mathbf{v}_1 = (-2, 0, 1)$.
3. Look at

$$(A - I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

   to find generalized eigenvector $\mathbf{v}_2 = (0, 1, 0)$.
4. Finally, $(A - I)^3 = \mathbf{0}$, so we get $\mathbf{v}_3 = (1, 0, 0)$.

Facts about generalized eigenvectors

The aim of generalized eigenvectors was to enlarge a set of linearly independent eigenvectors to make a basis. Are there always enough generalized eigenvectors to do so?

Fact

If λ is an eigenvalue of A with algebraic multiplicity k, then

$$\text{nullity}\left((A - \lambda I)^k\right) = k.$$

In other words, there are k linearly independent generalized eigenvectors for λ.

Corollary

If A is an $n \times n$ matrix, then there is a basis for $\mathbb{R}^n$ consisting of generalized eigenvectors of A.

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. From last time, we have eigenvalue $\lambda = 1$ and eigenvector $\mathbf{v}_1 = (-2, 0, 1)$.
2. Solve $(A - I)\mathbf{v}_2 = \mathbf{v}_1$ to get $\mathbf{v}_2 = (0, -1, 0)$.
3. Solve $(A - I)\mathbf{v}_3 = \mathbf{v}_2$ to get $\mathbf{v}_3 = (-1, 0, 0)$.

Chains of generalized eigenvectors

Let A be an $n \times n$ matrix and $\mathbf{v}$ a generalized eigenvector of A corresponding to the eigenvalue λ. This means that

$$(A - \lambda I)^p \, \mathbf{v} = \mathbf{0}$$

for a positive integer p.

If $0 \leq q < p$, then

$$(A - \lambda I)^{p-q} \, (A - \lambda I)^q \, \mathbf{v} = \mathbf{0}.$$

That is, $(A - \lambda I)^q \mathbf{v}$ is also a generalized eigenvector corresponding to λ for $q = 0, 1, \ldots, p-1$.

Definition

If p is the smallest positive integer such that $(A - \lambda I)^p \, \mathbf{v} = \mathbf{0}$, then the sequence

$$(A - \lambda I)^{p-1} \, \mathbf{v}, \;\; (A - \lambda I)^{p-2} \, \mathbf{v}, \;\; \ldots, \;\; (A - \lambda I) \, \mathbf{v}, \;\; \mathbf{v}$$

is called a **chain** or **cycle** of generalized eigenvectors. The integer p is called the **length** of the cycle.

Chains of generalized eigenvectors

Example

In the previous example,

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

and we found the chain

$$\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ (A - \lambda I)\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \ (A - \lambda I)^2\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Remark

The terminal vector in a chain is always an eigenvector.

Fact

The generalized eigenvectors in a chain are linearly independent.

Introduction to Jordan form

What's the analogue of diagonalization for defective matrices? That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are the linearly independent generalized eigenvectors of A occurring in chains, what does the matrix $S^{-1}AS$ look like, where $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$?

Suppose that $\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_k$ is a chain of generalized eigenvectors, so that $(A - \lambda I)\mathbf{v}_i = \mathbf{v}_{i-1}$ for $i > 1$ and $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}$. Then we have

$$A\mathbf{v}_i = \lambda \mathbf{v}_i + \mathbf{v}_{i-1} \text{ for } i > 1$$
$$\text{and } A\mathbf{v}_1 = \lambda \mathbf{v}_1.$$

Jordan blocks

The matrix for $T(\mathbf{x}) = A\mathbf{x}$ with respect to a basis consisting of a chain of generalized eigenvectors will be a Jordan block:

Definition

If λ is a real number, then the square matrix of the form

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

is called a **Jordan block corresponding to** λ.

Jordan canonical form

In general, we will need to find more than one chain of generalized eigenvectors in order to have enough for a basis. Each chain will be represented by a Jordan block.

Definition

A square matrix consisting of Jordan blocks centered along the main diagonal and zeros elsewhere is said to be in **Jordan canonical form** (JCF).

Theorem

If S is the matrix whose columns are a basis of generalized eigenvectors of A arranged in chains, then $S^{-1}AS$ is a matrix in JCF. It is unique up to a rearrangement of the Jordan blocks.

We may therefore refer to this matrix as *the* Jordan canonical form of A, and we see that every matrix is similar to a matrix in JCF.

Examples

- The matrix

$$\begin{bmatrix} 2 & 1 & 0 & & & & & & \\ 0 & 2 & 1 & & & & & & \\ 0 & 0 & 2 & & & & & & \\ & & & 5 & 1 & & & & \\ & & & 0 & 5 & & & & \\ & & & & & 7 & 1 & & \\ & & & & & 0 & 7 & & \\ & & & & & & & 7 & \\ & & & & & & & & 9 \end{bmatrix}$$

  is in JCF. It contains five Jordan blocks.
- Any diagonal matrix is in JCF. All of its Jordan blocks are 1×1.
- The matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  is in JCF. It has two blocks of sizes 2 and 1.

Uses of JCF

Theorem

Two $n \times n$ matrices are similar if and only if they have the same Jordan canonical form (up to a rearrangement of the Jordan blocks).

Our main use for JCF will be solving $\mathbf{x}' = A\mathbf{x}$ when the matrix A is defective.