Oscillations of Mechanical Systems

Math 240 — Calculus III

Summer 2015, Session II

Thursday, July 30, 2015

Agenda

1. Free oscillation
   - No damping
   - Damping
2. Forced oscillation
   - No damping
   - Damping

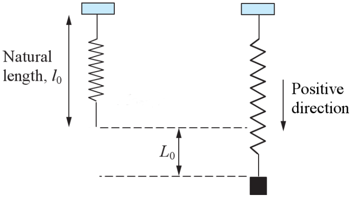
Introduction

We have now learned how to solve constant-coefficient linear differential equations of the form $P(D)y = F$ for a sizeable class of functions F.

We are going to use this knowledge to study the motion of mechanical systems consisting of a mass attached to a spring. Let's begin by modeling our system using a differential equation.

The setup

A mass of m kg is attached to the end of a spring with spring constant k N/m whose natural length is l_0 m. At equilibrium, the mass hangs without moving at a displacement of L_0 m, so that $mg = kL_0$.

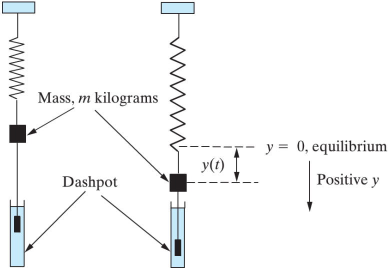


Spring-mass system in static equilibrium.

The setup

To analyze the system in motion, we let $y(t)$ denote the position of the mass at time t and take $y = 0$ to coincide with the equilibrium position. The forces that act on the mass are

1. The force of gravity, $F_g = mg$.
2. The spring force, $F_s = -k(y(t) + L_0)$.
3. A damping force proportional to the velocity of the mass, $F_d = -c\frac{dy}{dt}$.
4. Any external driving force, $F(t)$.



A damped spring-mass system.

Our equation

Newton says, the equation governing motion of the mass is

$$m\frac{d^2y}{dt^2} = F_g + F_s + F_d + F(t)$$

$$= mg - k(L_0 + y) - c\frac{dy}{dt} + F(t).$$

Rearranging gives us the linear differential equation

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t).$$

We may also have initial conditions

$$y(0) = y_0 \text{ and } y'(0) = v_0.$$

These indicate that at $t = 0$ the mass is displaced a distance of y_0 m and released with a downward velocity of v_0 m/s.

Free oscillation

First consider the case where there are no external forces acting on the system, that is, set $F(t) = 0$. Our differential equation reduces to

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0.$$

We will study the two subcases

- no damping: $c = 0$ and
- damping: $c > 0$.

No damping

Setting $c = 0$ in our equation, we have

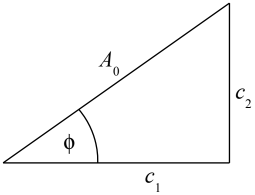
$$y'' + \omega_0^2 y = 0,$$

where $\omega_0 = \sqrt{k/m}$. This equation has the general solution

$$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

From the constants c_1 and c_2 we can derive

- the **amplitude**, $A_0 = \sqrt{c_1^2 + c_2^2}$,
- the **phase**, $\phi = \arctan(c_2/c_1)$.

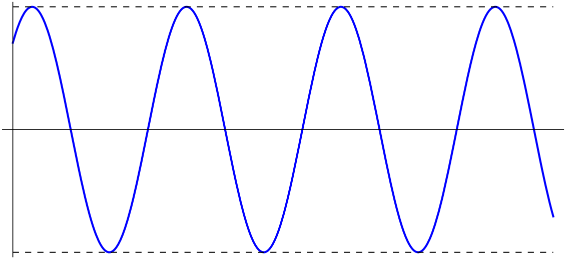


Using these new constants, the equation of our motion is

$$y(t) = A_0 \cos(\omega_0 t - \phi).$$

This is **simple harmonic motion**. The constant ω_0 is called the **circular frequency**.

Simple harmonic motion



This function is periodic with a **period** of $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$.

Its **frequency** is $f = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

Note that these quantities are independent of the initial conditions. They are properties of the system itself.

Damped motion

The motion of the system is damped when $c > 0$. Our equation is then

$$y'' + \frac{c}{m}y' + \frac{k}{m}.$$

The auxiliary polynomial has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

The behavior of the system will depend on whether there are distinct real roots, a repeated real root, or complex conjugate roots. This can be determined using the (dimensionless) quantity $c^2/(4km)$. We say that the system is

- *underdamped* if $c^2/(4km) < 1$ (complex conjugate roots),
- *critically damped* if $c^2/(4km) = 1$ (repeated real root),
- *overdamped* if $c^2/(4km) > 1$ (distinct real roots).

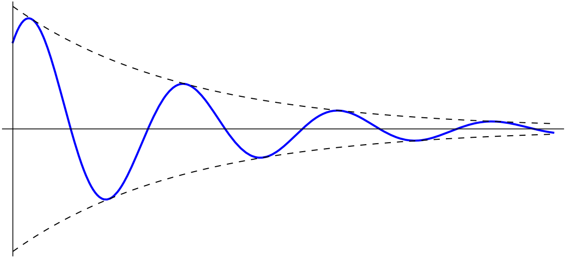
Underdamping

The two complex roots of the auxiliary polynomial give rise to the general solution

$$y(t) = e^{-ct/(2m)}(c_1 \cos \mu t + c_2 \sin \mu t),$$

where $\mu = \sqrt{4km - c^2}/(2m)$. Using amplitude and phase, it's

$$y(t) = A_0 e^{-ct/(2m)} \cos(\mu t - \phi).$$



Although the amplitude decays exponentially, this motion has a constant **quasiperiod** $T = \frac{2\pi}{\mu} = \frac{4\pi m}{\sqrt{4km - c^2}}$.

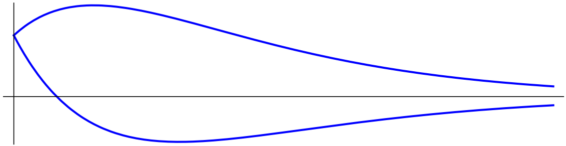
Critical damping

Critical damping happens when $c^2/(4km) = 1$. Then the equation

$$y'' + \frac{c}{m}y' + \frac{c^2}{4m^2}y = 0$$

has general solution

$$y(t) = e^{-ct/(2m)}(c_1 + c_2 t).$$



The motion is not oscillatory—it will pass through $y = 0$ at most once.

Overdamping

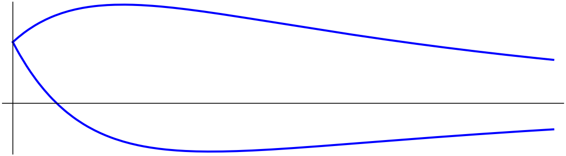
When $c^2/(4km) > 1$ we have two real roots of the auxiliary polynomial:

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} \text{ and } r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

Thus, our general solution is

$$y(t) = e^{-ct/(2m)}(c_1 e^{\mu t} + c_2 e^{-\mu t}),$$

where $\mu = \sqrt{c^2 - 4km}/(2m)$.



Overdamped motion is qualitatively similar to critically damped—it is not oscillatory and passes through the equilibrium position at most once.

Sanity check

Notice that in all three cases of damped motion, the amplitude diminishes to zero as $t \to \infty$. This is certainly what we expect in such a system.

Forced oscillation

Let's investigate the nonhomogeneous situation when an external force acts on the spring-mass system. We will focus on periodic applied force, of the form

$$F(t) = F_0 \cos \omega t,$$

for constants F_0 and ω. Our general equation is now

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{F_0}{m} \cos \omega t.$$

No damping

Setting $c = 0$, we want to solve

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega t,$$

where again $\omega_0 = \sqrt{k/m}$ is the circular frequency. We have seen that the complementary function is

$$y_c(t) = A_0\cos(\omega_0 t - \phi).$$

Our trial solution will depend on whether $\omega = \omega_0$.

Forced harmonic oscillation

When $\omega \neq \omega_0$, we can use the trial solution

$$y_p(t) = A\cos\omega t + B\sin\omega t.$$

Then we can find the particular solution

$$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t,$$

so the general solution is

$$y(t) = A_0\cos(\omega_0 t - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t.$$

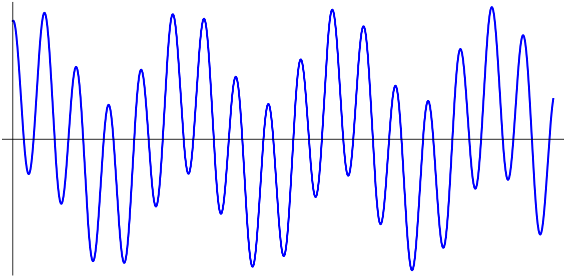
From this we see that the motion will look like a superposition of two simple harmonic oscillations.

Forced harmonic oscillation

If ω/ω_0 is a rational number, say, p/q, then the resulting motion will have a period of

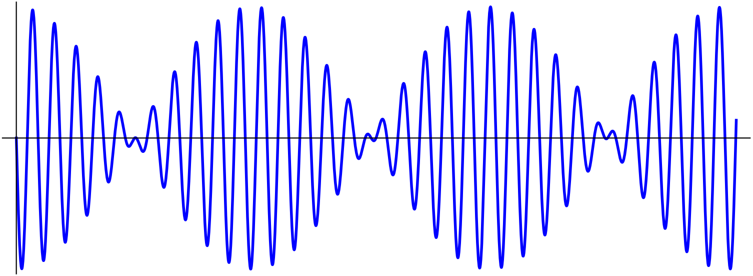
$$T = \frac{2\pi q}{\omega_0} = \frac{2\pi p}{\omega}.$$

Otherwise, it will be oscillatory but not periodic.



Forced harmonic oscillation

Interesting things happen when ω is very close to ω_0:



Resonance

If instead we have $\omega = \omega_0$, then we must use the trial solution

$$y_p(t) = t(A\cos\omega_0 t + B\sin\omega_0 t).$$

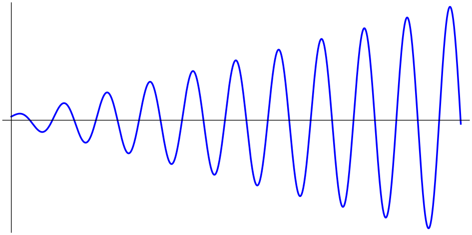
This leads to the particular solution

$$y_p(t) = \frac{F_0}{2m\omega_0} t\sin\omega_0 t$$

and general solution

$$y(t) = A_0\cos(\omega_0 t - \phi) + \frac{F_0}{2m\omega_0} t\sin\omega_0 t.$$

Notice that the amplitude increases without bound as $t \to \infty$.



Damping

Finally, we will consider a damped nonhomogeneous equation

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_0}{m}\cos\omega t,$$

with $c > 0$. The trial solution $y_p(t) = A\cos\omega t + B\sin\omega t$ yields the particular solution

$$y_p(t) = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2}\left[(k - m\omega^2)\cos\omega t + c\omega\sin\omega t\right].$$

Letting

$$H = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \text{ and } \eta = \arctan\left(\frac{c\omega}{m(\omega_0^2 - \omega^2)}\right)$$

turns it into

$$y_p(t) = \frac{F_0}{H}\cos(\omega t - \eta).$$

Damping

As before, the system can be underdamped, critically damped, or overdamped. Which one will determine the complementary function.

In each case of damped harmonic motion, the amplitude dies out as t gets large. But the driving force has a constant amplitude and thus it will dominate. We therefore refer to the complementary function as the **transient** part of the solution and call y_p the **steady-state** solution.