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In this talk we will introduce a construction called family algebras for the purpose of computing G-harmonic polynomials.
We start with the construction of the usual harmonic polynomials.
For a vector space U with an inner product g_{ab} on it, we have a Laplacian Δ that acts on the polynomials on U. We can define the Laplacian just from the inner product, and so the Laplacian is invariant under $SO(U)$. As a result, the set of polynomials $\mathcal{H}(U) = \{f \in S(U) | \Delta f = 0\}$ is also invariant under $SO(U)$, and thus has a variety of nice geometric, algebraic, and analytic interpretations, and is amenable to combinatorial and representation-theoretic methods.
We have the nice statement that

$$S(U) = \mathbb{C}[g_{ab}x^a x^b] \otimes \mathcal{H}(U)$$

for a basis x^a of U, so since $g_{ab}x^a x^b$ is invariant under $SO(U)$, all of the $SO(U)$ structure of $S(U)$ is contained in $\mathcal{H}(U)$.
We can generalize this to other representations and other groups. Given a group G and a representation U of G, we set $S(U)$ to be the polynomials on $U^\vee$. Set $I(U)$ to be $S(U)^G$, and ask when $S(U)$ is a free $I(U)$-module.
Not all groups have such representations; for instance, if G is finite, then by Chevalley and Shepard-Todd we have that G must be a finite pseudoreflection group and U must be the representation on which G acts by pseudoreflections.
For G a simple Lie group, we have by Kostant and Rallis that any symmetric space representation U of G has $S(U)$ a free module over $I(U)$.
In these cases we can define a space of harmonic polynomials. We define Di as an algebra map sends x^a to $\frac{\partial}{\partial x_a}$, where $\{x_a\}$ is the dual basis to a; this takes $I(U)$ to a space $D(U)$ of G-invariant differential operators that generalize the Laplacian. We restrict to $D_+(U)$, the elements of $D(U)$ that vanish on constant polynomials, and define a G-harmonic polynomial in $S(U)$ to be a polynomial annihilated by all elements of $D_+(U)$. We write it as $\mathcal{H}(G,U)$, unless G is understood, in which case we just write $\mathcal{H}(U)$.
In these cases,

$$S(U) = I(U) \otimes \mathcal{H}(U)$$

as graded G-modules.
Given such a setup, we wish to find the decomposition of $\mathcal{H}(U)$ into irreducible components. In particular, we define the graded multiplicity of a G-module V to be

$$m^V_{\mathcal{H}U}(q) = \sum_k \dim Hom_G(V, \mathcal{H}^k(U))q^k$$

In the case of G a pseudoreflection group, the degrees in the graded multiplicities are called fake degrees and are fully known for the irreducible representations of the pseudoreflection groups by the work of Lusztig, Benson and Surowski.
For G a simple Lie group, $U = \mathfrak{g}$ (the adjoint representation), the degrees in graded multiplicity are called generalized exponents.
Here the harmonic polynomials have an interpretation as harmonic functions on G and on the flag manifolds you get from quotients of G, so there's a nice connection to harmonic analysis on manifolds. For instance, if we let $V = \mathfrak{g}$, we recover what are called the exponents of G, thus the name generalized exponents. Note that the exponents tell us the structure of $I(\mathfrak{g})$.
There's no general closed form for the graded multiplicities in this case. Hesselink showed that the graded multiplicities can be computed as a q-analogue of the 0-weight multiplicity in V, which is computable via q-analogues of the Kostant or Fruedenthal multiplicity formulas, but as unwieldy as these formulas can be in the usual case, the q-analogues are even more difficult to deal with.
For the case of $G = Sl(n)$, and thus $U = sl(n)$, the graded multiplicities are the Kostka polynomials, which show up in a number of combinatorial statements.
Another known thing is Broer's theorem: for V a small representation,

$$m^V_{\mathcal{H}(G,\mathfrak{g})}(q) = \sum_i m^{U_i}_{\mathcal{H}(W,\mathfrak{h})}(q)$$

where $V^T = \bigoplus U_i$ as W-modules, and $\mathfrak{h}$ is the Cartan subalgebra. Unfortunately, small here is very restrictive, in that we need V to have no weights equal to twice a root of G. Thus Broer's theorem only applies to a fairly short list of representations. For example, it includes the adjoint representation, but that's about as big as it can get.
The main computational issue is that $Hom_G(V, \mathcal{H}(\mathfrak{g}))$ doesn't have much structure in general. It's a vector space but it doesn't even have a module structure beyond that, since we've already used the G-equivariance.
So we try to build something with more to it.
For a representation V of G, we define family algebras by

$$C_V(\mathfrak{g}) = Hom_G(End(V), S(\mathfrak{g}))$$

By Kostant, $C_V(\mathfrak{g})$ is a free $I(\mathfrak{g})$-module, and so we can ask for a basis of $C_V(\mathfrak{g})$ over $I(\mathfrak{g})$.
We can write $End(V) = V \otimes V^\vee = \bigoplus_i V_i$; we call the V_i the children of V and $V^\vee$.

$$\begin{aligned} C_V(\mathfrak{g}) &= Hom_G(End(V), S(\mathfrak{g})) = \bigoplus_i Hom_G(V_i, S(\mathfrak{g})) \quad (0.1) \\ &= \bigoplus_i Hom_G(V_i, \mathcal{H}(\mathfrak{g})) \otimes I(\mathfrak{g}) \quad (0.2) \end{aligned}$$

Hence the name family algebras.
So we get that a basis of the $Hom_G(V_i, \mathcal{H}(\mathfrak{g}))$ gives a basis for $C_V(\mathfrak{g})$ over $I(\mathfrak{g})$, and vice-versa.
Moreover, $C_V(\mathfrak{g})$ is an algebra, in that $Hom_G(End(V), S(\mathfrak{g})) = (End(V) \otimes S(\mathfrak{g}))^G$, so

we can use the multiplicative structures on $End(V)$ and $S(\mathfrak{g})$. The hope is then that, since we have this multiplicative structure, we can use that to determine an $I(\mathfrak{g})$-basis for $C_V(\mathfrak{g})$ of homogeneous elements, and from there get bases for $Hom_G(V_i, \mathcal{H}^k(\mathfrak{g}))$. Oddly enough, though, the Hopf structures don't seem to play well with the G-invariance condition. $End(V)$ has a coproduct and $S(\mathfrak{g})$ has a coproduct, but for elements of $(End(V) \otimes S(\mathfrak{g}))^G$, the coproduct ends up in the G-invariant part of

$$(End(V) \otimes S(\mathfrak{g})) \otimes (End(V) \otimes S(\mathfrak{g}))$$

rather than in the much smaller subspace

$$(End(V) \otimes S(\mathfrak{g}))^G \otimes (End(V) \otimes S(\mathfrak{g}))^G$$

So what is known here?
As one might expect, the Weyl group plays an important role.
We take the restriction map mentioned in Broer's theorem and apply it to $C_V(\mathfrak{g})$. The image ends up in

$$B_V(\mathfrak{h})^W = Hom_W(End(V)^T, S(\mathfrak{h}))$$

The restriction map is an injection of algebras here. Indeed, tensoring $C_V(\mathfrak{g})$ and $B_V(\mathfrak{h})^W$ with $Frac(I(\mathfrak{g})) \cong Frac(I(\mathfrak{h}))$ makes the restriction map into an isomorphism, but without that localization we only get an isomorphism if all of the children of V are small.
If we tensor further with the fraction field of $S(\mathfrak{h})$ we get an isomorphism with

$$\bigoplus_{\lambda \in Wt(V)} Mat_{m_V^\lambda}(FracS(\mathfrak{h}))$$

So we can read $B_V(\mathfrak{h})^W$ as a subalgebra of a sum of matrix algebras, and $C_V(\mathfrak{g})$ as a subalgebra of $B_V(\mathfrak{h})^W$.
Given an element P of $I(\mathfrak{g})$, can define an element

$$M_P = \sum_\alpha \pi(X_\alpha) \otimes \frac{\partial}{\partial X_\alpha} P \in C_V(\mathfrak{g})$$

When P is the quadratic Casimir element given by the Killing form, we just write $M_P = M$.
We say that a representation V has simple spectrum if each weight of V has multiplicity 1. Using the restriction map above, Rozhkovskaia shows that V has simple spectrum if and only if the $C_V(\mathfrak{g})$ is commutative, and indeed $C_V(\mathfrak{g}) \otimes_{I(\mathfrak{g})} Frac(I(\mathfrak{g}))$ generated over $Frac(I(\mathfrak{g}))$ by the element M. Unfortunately, having simple spectrum is even more restrictive than being small. For G not $sl(2)$, even the adjoint representation doesn't have simple spectrum. F_4 and E_8 have no representations with simple spectrum at all; their minimal representations have 0-weight spaces of dimension 2 and 8 respectively.
For representations with simple spectrum, while $C_V(\mathfrak{g})$ is generated over $Frac(I(\mathfrak{g}))$ by M, the same is not true if we only look at elements over $I(\mathfrak{g})$. For A_r, B_r, C_r and G_2, the defining representations have simple spectrum and the corresponding family algebras are all generated over $I(\mathfrak{g})$ by M, but for the family algebra of the defining

representation of D_r we need 2 generators, M and M_{Pf} where Pf is the Pfaffian. For the 27-dimensional representation of E_6 and the 56-dimensional representation of E_7, the family algebras need 3 generators over $I(\mathfrak{g})$, and for the spin representations of B_r and D_r we need roughly $r/2$ generators, all of which can be written as M_P for various $P \in I(\mathfrak{g})$.

For instance, for the $2r$-dimensional representation of D_r, restricting to the Cartan subalgebra gives

$$M = \begin{bmatrix} 0 & h_1 & \cdots & 0 & 0 \\ -h_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_r \\ 0 & 0 & \cdots & -h_r & 0 \end{bmatrix}$$

$$M_{Pf} = \begin{bmatrix} 0 & h_2h_3\cdots h_r & \cdots & 0 & 0 \\ -h_2h_3\cdots h_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_1h_2\cdots h_{r-1} \\ 0 & 0 & \cdots & -h_1h_2\cdots h_{r-1} & 0 \end{bmatrix}$$

We get relations in which M^{2r-3} decomposes into smaller-degree elements, as do M_{Pf}^2 and MM_{Pf}. So we get with $2r$ $I(\mathfrak{g})$-linearly independent elements in degrees $0, 1, \ldots, 2r-2, r-1$.

$$V_{\omega_1} \otimes V_{\omega_1}^\vee = V_0 \oplus V_{\omega_2} \oplus V_{2\omega_1}$$

and we already know the graded multiplicities for V_0, the trivial representation, and V_{ω_2}, the adjoint representation. But we can also use projections from $V_{\omega_1} \otimes V_{\omega_1}^\vee$ into the various subspaces to get the graded multiplicities. We end up with

$$\begin{aligned} m^{V_0}_{\mathcal{H}(\mathfrak{g})}(q) &= 1 \\ m^{V_{\omega_2}}_{\mathcal{H}(\mathfrak{g})}(q) &= q + q^3 + \ldots + q^{2r-3} + q^{r-1} \\ m^{V_{2\omega_1}}_{\mathcal{H}(\mathfrak{g})}(q) &= q^2 + q^4 + \ldots + q^{2r-2} \end{aligned}$$

In comparison to the peculiarities of the small representations, the $I(\mathfrak{g})$-bases for $C_V(\mathfrak{g})$ where V is the adjoint representation obey a nice pattern depending only on exponents of G:

$$\begin{aligned} &M^m R_n \text{ for } 0 \le m \le e_r + 1, 1 \le n \le r \\ &R_m S R_n + R_n S R_m \text{ for } 1 \le m \le n < r \\ &R_m S R_n - R_n S R_m \text{ for } 1 \le m < n \le r \end{aligned}$$

where M has degree 1, S has degree 2, and R_n has degree $e_n - 1$. The set of generators does depend on the group; A_r, B_r, C_r and G_2 all need only M, S and R_2, while D_r needs M, S, R_2 and R_r and F_4, E_6, E_7 and E_8 need M, S, R_2 and R_3.

For the classical groups the elements of tensor algebras $\bigotimes^* \mathfrak{g}$ can be described in terms of products of traces over various representations, and so the elements of the family algebras for tensor powers of the adjoint representation can be similarly described, giving spanning and generating sets. Similarly, for each classical series the decomposition of tensor powers of the adjoint representation become stable in the Church-Farb sense, so the projection operators are fairly uniform; thus the only remaining issue is turning the spanning set into a basis.

The exceptionals as always are a little more finicky in terms of the tensor algebra, as the defining representations have higher-degree invariants, but at least in the case of the tensor square of the adjoint representation they also admit a uniform-ish decomposition.

There are also several not-well-explored directions.

For instance, as mentioned the theorem of Kostant and Rallis applies to all symmetric space representations, so we could look at $C_V(U) = (End(V) \otimes S(U))^G$ and get graded multiplicities for V_i as components of $S(U)$.

We can also look at behaviour in positive characteristic; Willembring and Wallach showed that for good primes, the Kostant-Rallis theorem still applies.

Kirillov also defines a quantum family algebra $Q_V(\mathfrak{g})$, using $U(\mathfrak{g})$ instead of $S(\mathfrak{g})$. By the PBW theorem $S(\mathfrak{g})$ is isomorphic to $I(\mathfrak{g})$ as a G-module, with $I(\mathfrak{g})$ being isomorphic to $Z(\mathfrak{g}) = U(\mathfrak{g})^G$, so as $I(\mathfrak{g}) \cong Y(\mathfrak{g})$-modules, $C_V(\mathfrak{g})$ and $Q_V(\mathfrak{g})$ are isomorphic. But the multiplication is different.

We get things like $Q_V(\mathfrak{g})$ being commutative iff V has simple spectrum, again by Rozhkovskaia. We also get that for an element A of $Q_V(\mathfrak{g})$, there is a degree $\dim(V)$ polynomial P with coefficients in $Z(\mathfrak{g})$ such that $P(A) = 0$, but the coefficients for P cannot be read off as polynomials of the elements of A the way they can for matrices over commutative rings. So we get new variants on the trace and determinant. We can write $sl(2)$ as having three generators, H, E and F subject to

$$[H,E] = 2E, [H,F] = -2F, [E,F] = H$$

In this case $I(\mathfrak{g})$ and $Y(\mathfrak{g})$ are both generated by a single element that can be written as

$$\Delta_2 = \frac{H^2}{2} + EF + FE$$

Then we can take the 2-dimensional representation and look at the matrix

$$\begin{bmatrix} \frac{H}{2} & F \\ E & -\frac{H}{2} \end{bmatrix}$$

which we can read as either a matrix in $C_V(\mathfrak{g})$ or in $Q_V(\mathfrak{g})$. Write it as M for the matrix interpreted as an element of $C_V(\mathfrak{g})$, and write $\tilde{M}$ for the matrix interpreted as an element of $Q_V(\mathfrak{g})$.

Squaring the matrix gives

$$\begin{bmatrix} \frac{H^2}{4} + FE & \frac{H}{2}F - F\frac{H}{2} \\ E\frac{H}{2} - \frac{H}{2}E & \frac{H^2}{4} + EF \end{bmatrix} = \begin{bmatrix} \frac{H^2}{4} + \frac{EF+FE}{2} - \frac{[E,F]}{2} & \frac{[H,F]}{2} \\ \frac{[E,H]}{2} & \frac{H^2}{4} + \frac{EF+FE}{2} + \frac{[E,F]}{2} \end{bmatrix}$$

In $C_V(\mathfrak{g})$, the commutators vanish, leaving

$$M^2 = \begin{bmatrix} \frac{H^2}{4} + \frac{EF+FE}{2} & 0 \\ 0 & \frac{H^2}{4} + \frac{EF+FE}{2} \end{bmatrix} = \frac{\Delta_2}{2} Id$$

This gives us $M^2 - \frac{\Delta_2}{2} Id = 0$, and thus gives us a trace of 0 and a determinant of $-\frac{\Delta_2}{2}$, as we'd expect.
In $Q_V(\mathfrak{g})$, however, the commutators do not vanish. Instead we get

$$\tilde{M}^2 = \begin{bmatrix} \frac{H^2}{4} + \frac{EF+FE}{2} - \frac{H}{2} & -F \\ -E & \frac{H^2}{4} + \frac{EF+FE}{2} + \frac{H}{2} \end{bmatrix} = -\tilde{M} + \frac{\Delta_2}{2} Id$$

So we get $\tilde{M}^2 + \tilde{M} - \frac{\Delta_2}{2} Id = 0$, thus giving us a trace of -1.
In the classical Cayley-Hamilton construction, the trace can be computed as the sum of the diagonal elements, which is not possible here. Nor can we compute the trace from the trace of the corresponding element of $C_V(\mathfrak{g})$ using the isomorphisms of $I(\mathfrak{g})$ to $Y(\mathfrak{g})$. The same is true of the determinant in general; while for this case the determinant of $\tilde{M}$ mirrors that of M, this is not true in general,