Kirillov's Family Algebras and the Cayley-Hamilton Identity

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Harmonic Polynomials

Family Algebras

Known results

Other directions

Harmonic functions

$U = \mathbb{C}^n$, basis x_a, g^{ab} metric, dual metric g_{ab}.
$S(U)$ symmetric algebra on U, polynomials on $U^\vee$.

$$\Delta = g_{ab} \frac{\partial^2}{\partial x_a \partial x_b}$$

Harmonic polynomials

$$\mathcal{H}(U) = \{f \in S(U) | \Delta f = 0\}$$

Useful in analysis, geometry, algebra, interesting rep theoretic, combinatorial behavior.

SO(n) action

$SO(n)$ acts on U, $S(U)$, Δ.
Δ is $SO(n)$-invariant
$\mathcal{H}(U)$ is $SO(n)$-invariant; can look at as $SO(n)$-module.
Define $I(U) = S(U)^{SO(n)}$. $I(U) = \mathbb{C}[x_1^2 + x_2^2 + \ldots + x_n^2]$.

$$S(U) \cong I(U) \otimes \mathcal{H}(U)$$

$\mathcal{H}(U)$ encodes $SO(n)$-module behavior of $S(U)$.

Generalization

Group G, representation U.
When is $S(U)$ a free module over $I(U) = S(U)^G$?

Theorem (Sheppard-Todd, Chevalley)

For G finite, $S(U)$ is a free module over $I(U)$ iff G is a pseudoreflection group acting on U by pseudoreflections.

Theorem (Kostant-Rallis)

$S(U)$ is a free-module over $I(U)$ if G is a simple Lie group and U a symmetric space representation of G. Includes usual notion of harmonic.

G-Harmonic Polynomials

Basis x_a of U, g^{ab} a G-invariant bilinear form, dual g_{ab}
$Di : S(U) \to Diff(U)$, $x_a \to g_{ab}\frac{\partial}{\partial x_a}$.
$D(U) = Di(I(U))$, $D_+(U) = \{d \in D(U) | dc = 0 \text{ for } c \text{ constant}\}$

$$\mathcal{H}(G, U) = \{f \in S(U) | df = 0 \ \forall d \in D_+(U)\}$$

$\mathcal{H}(G, U)$ called G-harmonic polynomials. When understood, G omitted.

$$S(U) \cong I(U) \otimes \mathcal{H}(U)$$

Graded multiplicity

$I(U)$ G-invariant, again $\mathcal{H}(U)$ encodes G-module behavior of $S(U)$.
Define graded multiplicitiy:

$$m^{V}_{\mathcal{H}(U)}(q) = \sum_{k} \dim Hom_G(V, \mathcal{H}^k(U))q^k$$

For G pseudoreflection group, degrees called fake degrees; fully known.
For G simple Lie group, $U = \mathfrak{g}$ corresponding Lie algebra, degrees called generalized exponents.
$I(U)$ polynomial algebra in rank of U.

Generalized Exponents

For $V = \mathfrak{g}$, generalized exponents just regular exponents, $e_1, \ldots, e_r$.
Algebraic generators of $I(\mathfrak{g})$ in degrees $e_i + 1$.

Theorem (Hesselink)

$$m^V_{\mathcal{H}(\mathfrak{g})}(q) = m^0_V(q)$$

q-analogue of 0*-weight multiplicity in* V*.*

Can use Kostant, Fruedenthal multiplicity formulas; unwieldy.

More Generalized Exponents

Partially known for $G = SL(n)$; called Kostka polynomials.

Theorem (Broer)

Let T maximal torus in G, $\mathfrak{h}$ corresponding Cartan subalgebra, $W = N_G(T)/T$ Weyl group of G.
V^T admits W-action, define $\{U_i\}$ by $V^T = \bigoplus_i U_i$ as W-modules.
If V small, i.e. $2\lambda \notin Wt(V)$ for all roots λ,

$$m^V_{\mathcal{H}(G,\mathfrak{g})}(q) = \sum_i m^{U_i}_{\mathcal{H}(W,\mathfrak{h})}(q)$$

Not many small representations.

Family Algebras

$Hom_G(V, \mathcal{H}(\mathfrak{g}))$ vector space; no other structure.
Define classical family algebras

$$C_V(\mathfrak{g}) = Hom_G(End(V), S(\mathfrak{g}))$$

$End(V) = V \otimes V^\vee = \bigoplus_i V_i$, $\{V_i\}$ called children of V.

$$\begin{aligned} Hom_G(End(V), S(\mathfrak{g})) &= \bigoplus_i Hom_G(V_i, S(\mathfrak{g})) \\ &= \bigoplus_i Hom_G(V_i, \mathcal{H}(\mathfrak{g})) \otimes I(\mathfrak{g}) \end{aligned}$$

Homogeneous $I(\mathfrak{g})$-basis of $C_V(\mathfrak{g})$ gives homogeneous $\mathbb{C}$-basis of $Hom_G(V_i, \mathcal{H}(\mathfrak{g}))$ gives generalized exponents.

Multiplication

$$Hom_G(End(V), S(\mathfrak{g})) \cong (End(V) \otimes S(\mathfrak{g}))^G$$

Multiplicative structure. View $C_V(\mathfrak{g})$ as $I(\mathfrak{g})$-algebra. Idea: use multiplicative structure to describe $I(\mathfrak{g})$-module structure.
Hopf structures of $End(V)$, $S(\mathfrak{g})$ don't carry over. Naive coproduct of $A \in (End(V) \otimes S(\mathfrak{g}))^G$ in

$$(\ (End(V) \otimes S(\mathfrak{g})) \otimes (End(V) \otimes S(\mathfrak{g}))\ )^G$$

not in

$$(End(V) \otimes S(\mathfrak{g}))^G \otimes (End(V) \otimes S(\mathfrak{g}))^G$$

Restriction

Injection $Res : C_V(\mathfrak{g}) \to B_V(\mathfrak{h})^W = Hom_W(End(V)^T, S(\mathfrak{h}))$
Res isomorphism iff children of V small.

$$C_V(\mathfrak{g}) \otimes_{I(\mathfrak{g})} Frac(I(\mathfrak{g})) \cong B_V(\mathfrak{h})^W \otimes_{I(\mathfrak{h})} Frac(I(\mathfrak{g}))$$

$$C_V(\mathfrak{g}) \otimes_{I(\mathfrak{g})} Frac(S(\mathfrak{h})) \cong \sum_{\lambda \in Wt(V)} Mat_{m_V^\lambda}(Frac(S(\mathfrak{h})))$$

Generation

For $P \in I(\mathfrak{g})$,

$$M_P = \sum_\alpha \pi_V(X_\alpha) \otimes \frac{\partial}{\partial X_\alpha} P \in C_V(\mathfrak{g})$$

For P_2 quadratic Casimir element, write $M_{P_2} = M$.
V has simple spectrum if $m_V^\lambda \leq 1$ for all weights λ.

Theorem (Rozhkovskaia)

V has simple spectrum iff $C_V(\mathfrak{g}) \otimes_{I(\mathfrak{g})} Frac(I(\mathfrak{g}))$ is generated by M as $Frac(I(\mathfrak{g}))$-algebra iff $C_V(\mathfrak{g})$ is commutative.

Simple Spectrum

$C_V(\mathfrak{g})$ not necessarily generated by single element over $I(\mathfrak{g})$.
A_r, B_r, C_r, G_2, defining rep, $C_V(\mathfrak{g})$ generated by M
D_r, defining rep, $C_V(\mathfrak{g})$ generated by M, M_{Pf}
E_6, E_7, defining rep, $C_V(\mathfrak{g})$ three generators.
B_r, D_r, spin rep, $C_V(\mathfrak{g})$ (roughly) $r/2$ generators.

Example

D_r, defining $2r$-dimensional rep: Generators M in degree 1, M_{Pf} in degree $r-1$.
Relations: M^{2r} reduces, Cayley-Hamilton identity. $MM_{Pf} = Pf\ I_{2r}$.
M_{Pf}^2 reduces, M^{2r-1} also reduces.
$I(\mathfrak{g})$-basis: $1, M^n$ for $1 \leq n \leq 2r-2$, M_{Pf}. Degrees $0, 1, \ldots, 2r-2, r-1$.
$V_{\omega_1} \otimes V_{\omega_1} = V_0 \oplus V_{\omega_2} \oplus V_{2\omega_1}$

$$m_0^0(q) = 1$$

$$m_{\omega_2}^0(q) = q^1 + q^3 + \ldots + q^{2r-3} + q^{r-1}$$

$$m_{2\omega_1}^0(q) = q^2 + q^4 + \ldots + q^{2r-2}$$

D_r, $2r$-dimensional rep

$$M = \begin{bmatrix} 0 & h_1 & \dots & 0 & 0 \\ -h_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & h_r \\ 0 & 0 & \dots & -h_r & 0 \end{bmatrix}$$

$$M_{Pf} = \begin{bmatrix} 0 & h_2 h_3 \cdots h_r & \dots & 0 & 0 \\ -h_2 h_3 \cdots h_r & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & h_1 h_2 \cdots h_{r-1} \\ 0 & 0 & \dots & -h_1 h_2 \cdots h_{r-1} & 0 \end{bmatrix}$$

V = Adjoint Rep

$I(\mathfrak{g})$-basis of $C_{adj}(\mathfrak{g})$:

$$M^m R_n \text{ for } 0 \leq m \leq e_r + 1, 1 \leq n \leq r$$

$$R_m S R_n + R_n S R_m \text{ for } 1 \leq m \leq n < r$$

$$R_m S R_n - R_n S R_m \text{ for } 1 \leq m < n \leq r$$

$deg(M) = 1,\ deg(S) = 2,\ deg(R_n) = e_n - 1.$

Tensor powers of the adjoint rep

Trace invariant: given representation (M, π),

$$T_M(x_1, \dots, x_n) = tr_M(\pi(x_1)\pi(x_2)\cdots\pi(x_n))$$

Theorem

For a classical Lie group G, the invariant elements of $\bigotimes^* \mathfrak{g} \cong \bigotimes^* \mathfrak{g}^\vee$ are spanned by products of trace invariants.

For $V = \bigotimes^k \mathfrak{g}$, $C_V(\mathfrak{g})$ generated by elements based on trace invariants.

Symmetric Space Rep

U symmetric space rep

$$C_V(U) = Hom_G(End(V), S(U))$$

$$Hom_G(End(V), S(U)) = \bigoplus_i Hom_G(V_i, \mathcal{H}(U)) \otimes I(U)$$

Quantum Family Algebras

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

$$Q_V(\mathfrak{g}) = Hom_G(End(V), U(\mathfrak{g}))$$

Theorem (Poincare-Birkhoff-Witt)

$$S(\mathfrak{g}) \cong U(\mathfrak{g})$$

as G-modules.

Define $Z(\mathfrak{g}) = U(\mathfrak{g})^G$. $Z(\mathfrak{g}) \cong I(\mathfrak{g})$ by PBW. $Q_V(\mathfrak{g}) \cong C_V(\mathfrak{g})$

Cayley-Hamilton

Theorem (Kirillov)

For $A \in Q_V(\mathfrak{g})$, $\exists$ *monic* $P(x) \in Y(\mathfrak{g})[x]$, $deg(P) = n = \dim V$ *s.t.*

$$P(A) = 0$$

$P(x) = p_n + p_{n-1}x^{n-1} + \ldots + p_0$
$tr(A) = -p_{n-1}$, $det(A) = (-1)^n p_0$.
Not polynomials in entries of A.

Example

$$G = SL(2),\ V = \mathbb{C}^2,\ \Delta_2 = \frac{H^2}{2} + EF + FE$$

$$M = \begin{bmatrix} \frac{H}{2} & F \\ E & -\frac{H}{2} \end{bmatrix}$$

$$\begin{aligned} M^2 &= \begin{bmatrix} \frac{H^2}{4} + FE & \frac{H}{2}F - F\frac{H}{2} \\ E\frac{H}{2} - \frac{H}{2}E & \frac{H^2}{4} + EF \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left(\frac{H^2}{2} + EF + FE\right) - \frac{H}{2} & -F \\ -E & \frac{1}{2}\left(\frac{H^2}{2} + EF + FE\right) + \frac{H}{2} \end{bmatrix} \\ &= -M + \Delta_2 I_2 \end{aligned}$$