Final Exam

Name: SOLUTIONS

Math 103 - Introduction to Calculus
August 14, 2008

Exam Rules:

1. *No calculators of any sort are permitted on this exam. You may use one reference page prepared prior to the exam.*
2. *Justify your answer! A correct answer with no justification may receive no credit.*
3. *Clearly label your work.*

This exam consists of 10 questions, worth a total of 100 points.
Solutions appear in blue. Comments appear in magenta.

1. (12 points) Evaluate the integrals:

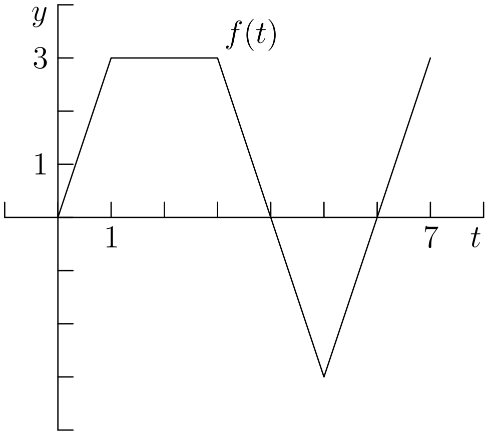
   (a) $\displaystyle\int_0^{\pi} \sin x \, dx = -\cos x\Big|_0^{\pi} = -\cos\pi + \cos 0 = 2$

   (b) $\displaystyle\int_1^{\sqrt{3}} x\left(x^2-1\right)^5 \, dx = \frac{1}{2}\int_0^2 u^5 \, du = \frac{1}{12}u^6\Big|_0^2 = \frac{16}{3}$
   (let $u = x^2 - 1$, so $du = 2x \, dx$)

   (c) $\displaystyle\int_0^{\pi} \sin x \cos^2 x \, dx = -\int_1^{-1} u^2 \, du = \int_{-1}^{1} u^2 \, du = \frac{1}{3}u^3\Big|_{-1}^{1} = \frac{2}{3}$
   (let $u = \cos x$, so $du = -\sin x \, dx$)

   If you plan to go on in calculus, it's *very* important that you know how to evaluate integrals such as these.

2. (12 points) Let $g(x) = \int_1^x f(t)\,dt$, where $f(t)$ is defined by the graph below. Note that the domain of g is $[0, 7]$.



(a) On what intervals is $g(x)$ increasing?

If $g(x) = \int_1^x f(t)\,dt$, then $g'(x) = f(x)$ by Fundamental Theorem of Calculus. Thus, $g(x)$ is increasing where $g'(x) = f(x)$ is positive, that is, on the intervals $0 < x < 4$ and $6 < x < 7$.

(b) Where does $g(x)$ have a local minimum?

$g(x)$ has a local minimum where its derivative, $f(x)$, changes from negative to positive. This happens at $x = 6$.

(c) Where is $g(x)$ concave up?

$g(x)$ is concave up where its second derivative is positive.
Since $g''(x) = f'(x)$, and $f'(x)$ is positive where $f(x)$ is increasing, we see that $g(x)$ is concave up where $f(x)$ is increasing.
Thus, $g''(x)$ is concave up on the intervals $0 < x < 1$ and $5 < x < 7$.

Realizing that $g'(x) = f(x)$ (g is the antiderivative of f) is key to this problem. In calculus, it's important to be able to look at a function and make conclusions about its derivative and antiderivative.

3. (8 points)

(a) Let g be a continuous function such that $\int_1^{27} g(u)\,du = -12$.

Find $\int_3^1 x^2 g(x^3)\,dx$.

Let $u = x^3$, so $du = 3x^2\,dx$.

Then:

$$\int_3^1 x^2 g(x^3)\,dx = \frac{1}{3}\int_{27}^1 g(u)\,du = \frac{-1}{3}\int_1^{27} g(u)\,du = \frac{-1}{3}(-12) = 4$$

Concepts involved here: u-substitution and properties of integrals.

(b) If $\frac{d}{dx}[f(3x)] = \sqrt{x}$, what is $f'(27)$?

$\frac{d}{dx}[f(3x)] = f'(3x)\cdot 3$, so $3f'(3x) = \sqrt{x}$.

That is, $f'(3x) = \frac{\sqrt{x}}{3}$.

Let $x = 9$, and we have $f'(27) = f'(3\cdot 9) = \frac{\sqrt{9}}{3} = \frac{3}{3} = 1$.

This is the **chain rule** again! Many people said that $f'(3x) = \sqrt{x}$, which is not correct.

4. (10 points) A box with a square base and open top is to have a volume of 32 cubic units. Find the dimensions that will minimize the amount of material used to make the box (i.e. minimize the surface area of the box).

Let x be the length of the side of the base of the box, and h be the height of the box.

The surface area of the box is the area of the bottom plus four times the area of one of the sides, so Area $= x^2 + 4xh$.

The volume of the box is $32 = x^2 h$, so $h = \dfrac{32}{x^2}$.

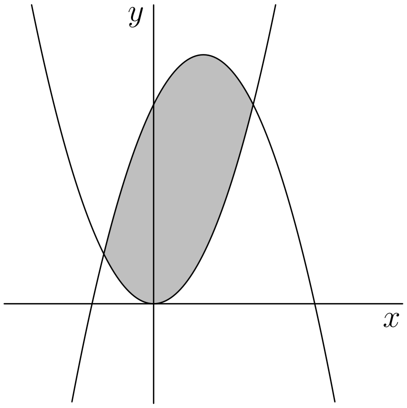
Thus, $A(x) = x^2 + 4x\dfrac{32}{x^2} = x^2 + \dfrac{128}{x}$.

Differentiating, $A'(x) = 2x - \dfrac{128}{x^2}$.

$A'(x) = 0$ when $2x = \dfrac{128}{x^2}$, or $x^3 = 64$, or $x = 4$.

Thus, the length of the side of the bottom of the box is $x = 4$, and the height of the box is $h = \dfrac{32}{4^2} = 2$.

5. (10 points) Find the area between the graphs of the functions $y = x^2$ and $y = -x^2 + 2x + 4$.



First find the points of intersection of the graphs:

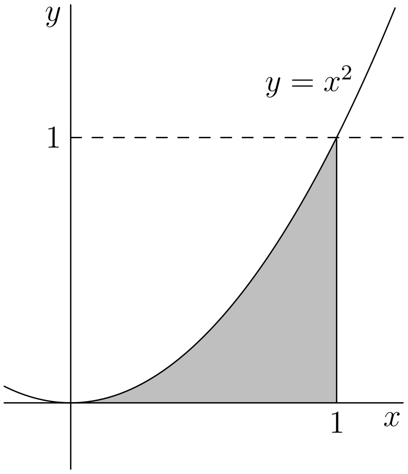
$$\begin{aligned} x^2 &= -x^2 + 2x + 4 \\ 0 &= 2x^2 - 2x - 4 \\ 0 &= x^2 - x - 2 \\ 0 &= (x-2)(x+1) \end{aligned}$$

So the intersections occur where $x = -1$ and $x = 2$. Note also that the upper graph is $y = -x^2 + 2x + 4$. Thus, the area between the graphs is:

$$\begin{aligned} \int_{-1}^{2} \left((-x^2 + 2x + 4) - x^2\right)\, dx &= \int_{-1}^{2} \left(-2x^2 + 2x + 4\right)\, dx \\ &= \left[\frac{-2}{3}x^3 + x^2 + 4x\right]_{-1}^{2} \\ &= \left(\frac{-16}{3} + 4 + 8\right) - \left(\frac{2}{3} + 1 - 4\right) \\ &= 9 \end{aligned}$$

Some people tried to find the area by integrating with respect to y (using horizontal rectangles), but that's a very difficult way to do this problem.

6. (10 points) The region bounded by the graph of $y = x^2$, $y = 0$, and $x = 1$ is rotated around the line $y = 1$. What is the volume of the resulting solid?



Slice vertically; slices are washers with outer radius 1 and inner radius $1 - x^2$. Thus, the volume of a representative washer is:

$$dV = \pi \left(1^2 - (1 - x^2)^2\right)\, dx = \pi \left(2x^2 - x^4\right)\, dx$$

The volume of the solid is then:

$$V = \pi \int_0^1 (2x^2 - x^4)\, dx = \pi \left[\frac{2}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi \left(\frac{2}{3} - \frac{1}{5}\right) = \frac{7\pi}{15}$$

7. (8 points)

This problem is testing your ability to apply the Fundamental Theorem of Calculus (version 1). Some people didn't realize this and did a lot of unnecessary work.

(a) If $\int_1^x f(t)\,dt = x^2$, what is $f(3)$?

Differentiating the given equation, we have $f(x) = 2x$ (by FTC 1).
Thus, $f(3) = 6$.

(b) If $h(x) = \int_{\sqrt{x}}^5 \frac{t}{t^2-1}\,dt$, what is $h'(3)$?

First observe:

$$h(x) = -\int_5^{\sqrt{x}} \frac{t}{t^2-1}\,dt$$

Now we can differentiate $h(x)$ using FTC 1:

$$h'(x) = -\frac{\sqrt{x}}{x-1} \cdot \frac{1}{2\sqrt{x}} = -\frac{1}{2x-2}$$

Thus, $h'(3) = \frac{-1}{4}$.

8. (8 points) Let $f(x) = 2x^2 + 6x + 9$. Find the line tangent to the graph of $f(x)$ with slope 2.

   $f'(x) = 4x + 6$, so $f'(x) = 2$ where $2 = 4x + 6$, or $x = -1$.

   At $x = -1$, $f(-1) = 2(1)^2 + 6(1) + 9 = 5$, so we want a line with slope 2 through the point $(-1, 5)$.

   The equation for such a line is $y - 5 = 2(x + 1)$, or $y = 2x + 7$.

   Note that we want the line to go through the point $(1, f(1))$, *not* the point $(1, f'(1))$. The derivative tells us the *slope* at the point of tangency, not the y-coordinate of the point itself.

9. (10 points) Let $g(x) = \dfrac{\sqrt{4x^6 + x + 1}}{x^3 + x^2 - 6x}$.

   (a) What are the horizontal asymptotes of $g(x)$?

   To find horizontal asymptotes, we investigate the limits of $g(x)$ as $x \to \pm\infty$.

   The denominator is degree 3, and the numerator is essentially degree 3 (its degree 6, but under a square root), so to find the limits we look at the ratio of the leading coefficients and the sign.

   The ratio is $\dfrac{\sqrt{4}}{1} = 2$. The numerator is always positive, but the denominator is negative as $x \to -\infty$, so the limits are:

$$\lim_{x\to\infty} g(x) = 2 \qquad \text{and} \qquad \lim_{x\to-\infty} g(x) = -2$$

   Thus, the horizontal asymptotes are $y = 2$ and $y = -2$.

   (b) What are the vertical asymptotes of $g(x)$?

   Vertical asymptotes occur where the denominator is zero and the numerator is not zero.

   The denominator factors as $x^3 + x^2 - 6x = x(x + 3)(x - 2)$, so it is zero when $x = -3$, 0, or 2. The numerator is not zero at these values, so vertical asymptotes occur at $x = -3$, $x = 0$, and $x = 2$.

10. (12 points) For each of the following, circle YES or NO, and **justify** your answer.

(a) Is the graph of $y^2 - 4xy + x^2 = -2$ concave up at the point $(1,1)$?

[✓ **YES**] **NO**

Differentiate implicitly:

$$2y\tfrac{dy}{dx} - 4\left(y + x\tfrac{dy}{dx}\right) + 2x = 0$$

$$\tfrac{dy}{dx} = \frac{2y - x}{y - 2x}$$

Differentiate again to find the second derivative:

$$\tfrac{d^2y}{dx^2} = \frac{(y-2x)\left(2\tfrac{dy}{dx} - 1\right) - (2y - x)\left(\tfrac{dy}{dx} - 2\right)}{(y-2x)^2}$$

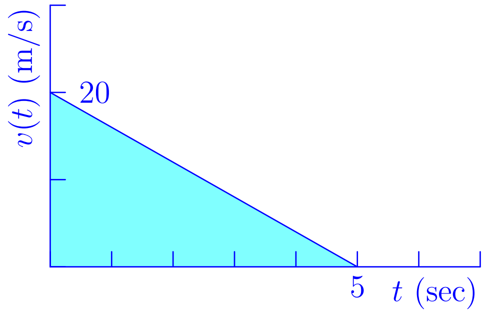
At the point $(1,1)$, $\tfrac{dy}{dx} = -1$, and so

$$\tfrac{d^2y}{dx^2} = \frac{(-1)(-2-1) - (1)(-3)}{(-1)^2} = 6$$

Since the second derivative is positive, the graph is concave up at the point $(1,1)$
You *must* consider the second derivative to make a conclusion about concavity!
You can't say that the function is concave up by simply looking at $\tfrac{dy}{dx}$ at a point.

(b) A car is traveling 20 m/s when the driver sees an obstacle on the road 45 m ahead and slams on the brakes. If the car decelerates at 4 m/s^2, will it stop in time to avoid a collision?

YES [✓ **NO**]



Let $t = 0$ be the moment the driver slams on the brakes. The velocity function is a line with slope -4 as pictured. The car will stop in 5 seconds. Distance traveled is the area under the velocity graph. The area under the velocity graph from $t = 0$ to $t = 5$ is 50, so the car will stop in 50 meters. Thus, the car does not quite stop within the 45 meters.

Some people had other valid solutions to this problem, but the one here is among the simplest.