Midterm 2

Name: SOLUTIONS

Math 103 - Introduction to Calculus
July 31, 2008

Exam Rules:

1. *No calculators of any sort are permitted on this exam. You may use one reference page prepared prior to the exam.*
2. *Justify your answer! A correct answer with no justification may receive no credit.*
3. *Clearly label your work.*

This exam consists of 10 questions, worth a total of 100 points.
Solutions appear in blue. Comments appear in magenta.

1. (12 points) Differentiate the following functions:

   (a) $y = \tan^2(3x)$

   $$y' = 2\tan(3x)\sec^2(3x)(3) = 6\tan(3x)\sec^2(3x)$$

   The most common error was forgetting the factor of $\tan(3x)$.

   (b) $y = x^3\sqrt{x^2-1}$

   $$y' = 3x^2\sqrt{x^2-1} + x^3\left(\tfrac{1}{2}\right)(x^2-1)^{-1/2}(2x) = 3x^2\sqrt{x^2-1} + \frac{x^4}{\sqrt{x^2-1}}$$

   (c) $y = x^2\cos\left(\dfrac{1}{x^2}\right)$

   $$y' = 2x\cos(x^{-2}) + x^2\left(-\sin(x^{-2})\right)\left(-2x^{-3}\right) = 2x\cos(x^{-2}) + \frac{2}{x}\sin(x^{-2})$$

   A common error in parts (b) and (c) was not applying the product rule. Knowing how to apply the product rule and chain rule is *essential* for success in calculus class, and it's not too late to learn!

2. (10 points) Consider the equation $y^3 - y - x^2 = -1$.

(a) Find an equation for the tangent line at $(1, 1)$.

We are given the point $(1, 1)$ through which the tangent line must pass; we must find the slope at that point. Differentiating implicitly:

$$3y^2 \tfrac{dy}{dx} - \tfrac{dy}{dx} - 2x = 0$$
$$\tfrac{dy}{dx}(3y^2 - 1) = 2x$$
$$\tfrac{dy}{dx} = \frac{2x}{3y^2 - 1}$$

So the slope at $(1, 1)$ is $\dfrac{2}{3-1} = 1$. Thus, the equation of the tangent line is $y - 1 = 1(x - 1)$, or simply $y = x$.

(b) Is the graph concave up or concave down at $(1, 1)$? How do you know?

To find the concavity, we need to look at the second derivative:

$$\tfrac{d^2y}{dx^2} = \frac{(3y^2 - 1)(2) - (2x)(6y\tfrac{dy}{dx})}{(3y^2 - 1)^2}$$
$$\tfrac{d^2y}{dx^2} = \frac{6y^2 - 2 - 12xy\frac{2x}{3y^2-1}}{(3y^2 - 1)^2}$$

So if $(x, y) = (1, 1)$, then $\tfrac{d^2y}{dx^2} = \dfrac{6 - 2 - 12}{2^2} = -2$. Since the second derivative is negative, the curve is concave down.

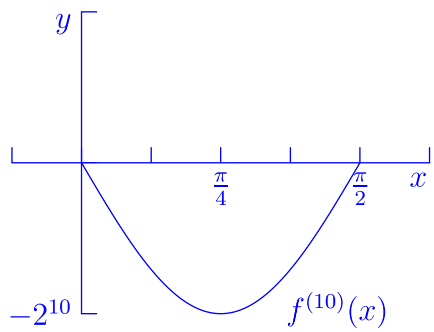
You *cannot* make a conclusion about concavity by simply knowing that the slope is positive. A function with positive slope may be either concave up or down. To test concavity, you *must* look at the sign of the second derivative.

3. (6 points) Let $f(x) = \sin(2x)$. What is the absolute maximum value attained by $f^{(10)}(x)$ on the interval $[0, \frac{\pi}{2}]$?

To find $f^{(10)}$, compute derivatives of f and look for a pattern:

$$f'(x) = 2\cos(2x) \qquad f''(x) = -2^2 \sin(2x) \qquad \cdots \qquad f^{(10)}(x) = -2^{10}\sin(2x)$$

On the interval $[0, \frac{\pi}{2}]$, the absolute maximum value of $\sin(2x)$ is zero. Thus, the absolute maximum value of $f^{(10)}$ is zero on $[0, \frac{\pi}{2}]$.



Some people wrote out ten derivatives of f, but you don't have to write that much if you see the pattern. You could use the Closed Interval Test to justify your answer, but then you must find the critical points of $f^{(10)}$, which are the zeros of $f^{(11)}$. A picture of $f^{(10)}$, like the one above, is also good justification.

4. (10 points) Linear approximation:

   (a) Use a linear approximation to estimate $\sqrt{104}$. Write your answer as a decimal.

   Let $f(x) = \sqrt{x}$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$.

   We know that $f(100) = 10$ and $f'(100) = \dfrac{1}{20}$.

   Near $x = 100$,

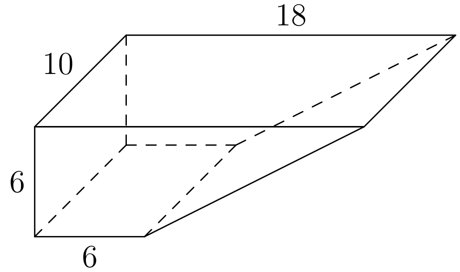
$$f(x) \approx f(100) + f'(100)(x - 100) = 10 + \frac{1}{20}(x - 100)$$

   Thus, $f(104) \approx 10 + \dfrac{1}{20}(104 - 100) = 10 + \dfrac{4}{20} = 10 + 0.2 = 10.2$

   (b) Is your answer an overestimate or an underestimate? Why?

   Since the graph of $\sqrt{x}$ is concave down, the tangent line is above the function and our estimate is too large.

5. (10 points) A pool 10 feet wide has a trapezoidal cross section, as shown below. The trapezoids have height 6 and bases 6 and 18 feet.



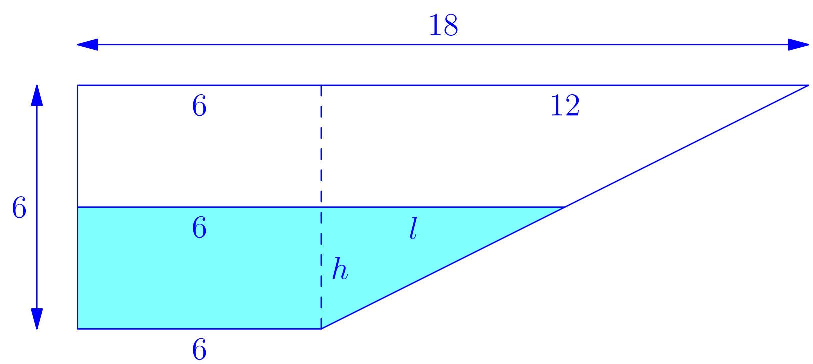
If the pool is being filled with water at a rate of 30 ft^3 per minute, how fast is the water level rising when the water is 3 feet deep?

Let V be the volume, and h be the depth, of water in the pool.

We know: $\frac{dV}{dt} = 30$ ft^3 per minute.

Want to find: $\frac{dh}{dt}$ when $h = 3$ ft.

Connecting equation: Let A be the area of the cross section of water in the pool. Then $V = 10A$ since the pool is 10 feet wide. To find A in terms of h, we use the following diagram of the cross section:



The shaded portion of the pool is filled with water, so we want to find the area of the shaded trapezoid. Its bottom side has length 6, its height is h, and the top side has length $h + l$ as pictured.

We need to find l in terms of h. By similar triangles, $\frac{12}{6} = \frac{l}{h}$, which implies $l = 2h$. Thus, the top side of the trapezoid has length $6 + 2h$.

The area of the trapezoid is then $\dfrac{6 + (6 + 2h)}{2}h = 6h + h^2$.

Therefore, the formula for the volume of water in the pool is:

$$V = (6h + h^2)10 = 60h + 10h^2$$

Differentiate: $\frac{dV}{dt} = 60\frac{dh}{dt} + 20h\frac{dh}{dt}$

Substitute and solve:

$$\frac{dV}{dt} = 60\frac{dh}{dt} + 20h\frac{dh}{dt}$$
$$30 = 60\frac{dh}{dt} + 20(3)\frac{dh}{dt}$$
$$30 = 120\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{4}$$

Therefore, the water level is rising at $\frac{1}{4}$ feet (or 3 inches) per minute.

Finding the connecting equation is the part of this problem that requires the most creative thinking. Yet it's important becuase mathematics is about problem solving, not simply memorizing formulas.

If your connecting equation was incorrect, it was possible to recieve most of the credit for this problem by having correct work based on your connecting equation.

6. (9 points) For each of the following, circle YES or NO, and briefly **justify** your answer.

(a) Does the graph of $f(x) = \frac{1}{2}x^2 - \cos x$ have a point of inflection?

YES **NO**

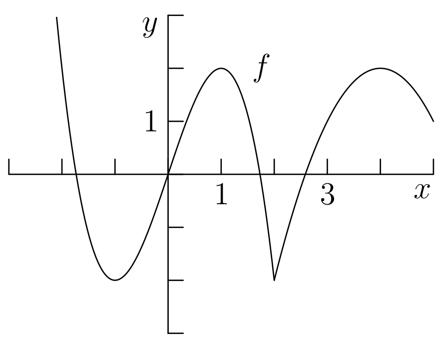
$$f'(x) = x + \sin x$$
$$f''(x) = 1 + \cos x$$

Notice that $f''(x) \geq 0$ for all x. Thus, the second derivative is never negative and cannot change sign, so there cannot be an inflection point.

Many people said that $f''(x) = 0$ for values such as $x = \pi$, and so f has inflection points. This is not true. Just as a critical point might be neither a local min nor max, a point where $f''(x) = 0$ might not be an inflection point. Inflection points occur only where the concavity *changes* from up do down, or from down to up.

(b) On the graph below, is the derivative increasing at $x = 3$?

YES **NO**



Since the graph is concave down at $x = 3$, the derivative is *decreasing*.

(c) If $f(x)$ is differentiable everywhere and $\lim_{x\to\infty} f'(x) = 1$, can you conclude that $\lim_{x\to\infty} f(x) = \infty$?

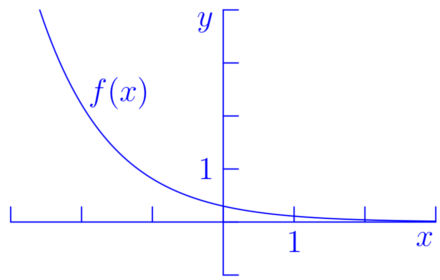
YES **NO**

Since $\lim_{x\to\infty} f'(x) = 1$, the graph of f is approaching a line with slope 1 as $x \to \infty$. All such lines go to infinity, so $\lim_{x\to\infty} f(x) = \infty$.

7. (8 points) For each of the following, either give an example or say why none exists. When giving an example, you may use an equation, graph, or other means of describing your function.

(a) A function f such that $f(x) > 0$ everywhere, $f''(x) > 0$ everywhere, and $\lim_{x\to\infty} f(x) = 0$.

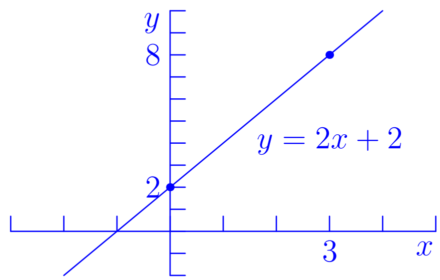
One example of such a function f is pictured here:



A function *can* have a horizontal asymptote and be always concave up!

(b) A function g such that $g(0) = 2$, $g(3) = 8$, and $g'(x) \leq 2$ everywhere.

The line $y = x + 2$ is one example of such a function.



This question is much like others we've seen where the Mean Value Theorem says that no function can satisfy certain criteria, but this time, a function does exist. Since the instructions say to give an example, it's not enough to simply say that the MVT doesn't prevent the function from existing.

8. (8 points) Suppose f is a function such that $f'(x) = \dfrac{1}{x}$.

(a) If $g(x) = f(x^3)$, what is $g'(3)$?

By the chain rule, $g'(x) = f'(x^3)(3x^2) = \dfrac{1}{x^3}(3x^2) = \dfrac{3}{x}$. Thus, $g'(3) = \dfrac{3}{3} = 1$.

(b) If $h(x) = f(3x)$, what is $h'(3)$?

Again by the chain rule, $h'(x) = f'(3x)(3) = \dfrac{1}{3x}(3) = \dfrac{1}{x}$. Thus, $h'(3) = \dfrac{1}{3}$.

The key for this problem is identifying that it involves composition of functions, and thus you need to use the chain rule.

9. (9 points) Find the limits, if they exist.

(a) $\lim_{x\to\infty} \frac{x^6 - 3x^9 + 15}{x^9 + 4x^2 + 3x} = -3$

Both the numerator and denominator have degree 9, so the limit is the ratio of the coefficients, which is $\frac{-3}{1} = -3$.

(b) $\lim_{x\to-\infty} \frac{\sqrt{x^6 - 4x^2 + 1}}{2x^3 + x - 3} = -\frac{1}{2}$

Both the numerator and denominator have degree 3 (in the numerator we have degree 6, but under a square root). The absolute value of the limit is the ratio of the coefficients, which is $\frac{1}{2}$. However, since the limit is as $x \to -\infty$, we notice that the numerator will be positive and the denominator will be negative, so the limit is $-\frac{1}{2}$.

(c) $\lim_{x\to\infty} \frac{(x+2)\sin x}{x^2 - 4} = \lim_{x\to\infty} \frac{\sin x}{x-2} = 0$

Since the numerator is bounded ($-1 \leq \sin x \leq 1$) and the denominator goes to infinity, the limit is 0.

10. (18 points) Let $f(x) = \dfrac{4-4x}{(x-2)^2}$.

(a) Find the x- and y-intercepts of the graph of $f(x)$.

To find the x-intercept, we solve $f(x) = 0$ and obtain $x = 1$.

For the y-intercept, we find $f(0) = \dfrac{4}{(-2)^2} = 1$.

(b) Find any asymptotes of $f(x)$.

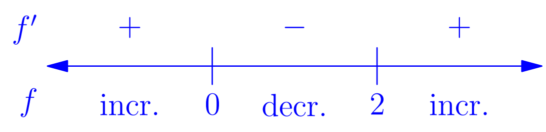
Vertical asymptotes occur where the denominator is zero but the numerator is not; in this case, at $x = 2$.

To find horizontal asymptotes, look at $\lim_{x\to\pm\infty} f(x)$. In this case, the horizontal asymptote is $y = 0$.

(c) Describe the intervals on which $f(x)$ is increasing and decreasing.

$$\begin{aligned} f'(x) &= \frac{(x-2)^2(-4) - (4-4x)(2)(x-2)}{(x-2)^4} \\ &= \frac{(x-2)(-4) - (4-4x)(2)}{(x-2)^3} = \frac{4x}{(2-x)^3} \end{aligned}$$

The critical numbers are $x = 0$ and $x = 2$.



From the sign chart, we see that the function is increasing on $(-\infty, 0)$ and $(2, \infty)$, and decreasing on $(0, 2)$.

Simplifying f' as much as possible makes it easier to find the critical numbers, and makes it *much* easier to find f'' and the inflection points in part (e).

(Problem 10, continued) Recall that $f(x) = \dfrac{4-4x}{(x-2)^2}$.

(d) Find all critical points and classify them as relative minima, relative maxima, or neither. You may use your work from part (c).

From part (c), the critical numbers are $x = 0$ and $x = 2$.

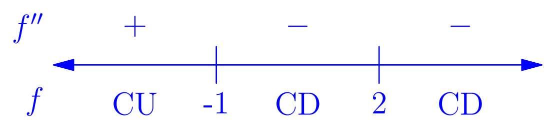
From the sign chart in part (c), we can apply the First Derivative Test to conclude that the function has a local max at $x = 0$.

We might be tempted to say that $x = 2$ is a local min, but from part (b) we know that it is in fact a vertical asymptote.

(e) Describe the concavity of $f(x)$.

$$f''(x) = \frac{(x-2)^3(4) - 4x(3)(x-2)^2}{(x-2)^6} = \frac{(x-2)(4) - 4x(3)}{(x-2)^4} = \frac{-8x-8}{(x-2)^4}$$

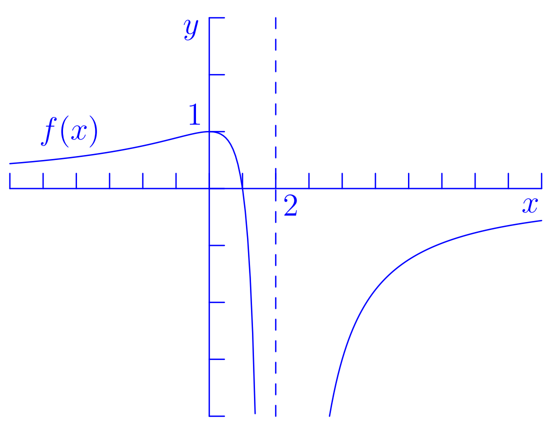
So the second derivative is zero when $x = -1$ and undefined when $x = 2$.



We conclude that $f(x)$ is concave up on the interval $(-\infty, -1)$ and concave down on the intervals $(-1, 2)$ and $(2, \infty)$.

We find an inflection point at $x = -1$, but $x = 2$ is not an inflection point.

(f) Sketch a graph of $f(x)$ using at least the information from parts (a) through (e). Be sure to label your graph. It is more important for your graph to reflect the information above than to have accurate scale.



If you made a mistake in this problem, you could receive partial credit if the work following the mistake was correct.