Math 104 Selected Homework Solutions, Sections 9.1-9.5

Section 9.1

30. We find the arc length of $y = \sqrt{4-x^2}$, $0 \le x \le 2$. Note that $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$. Thus,

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+\frac{x^2}{4-x^2}}\,dx = \int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx \\ &= \int_{\pi/2}^0 \frac{-4\sin\theta}{2\sin\theta} d\theta \qquad \leftarrow \text{use trig substitution } 2\cos\theta = x \\ &= \int_0^{\pi/2} 2\,d\theta = \pi \end{aligned}$$

We can check that our answer is correct by noting that the curve is a quarter circle of radius 2.

24. We use Simpson's Rule with $n = 10$ to estimate the arc length of $y = x\ln x$, $1 \le x \le 3$. The arc length is given by $L = \int_1^3 \sqrt{1+(1+\ln x)^2}\,dx$. Let $f(x) = \sqrt{1+(1+\ln x)^2}$. Then by Simpson's rule, the integral is:

$$\begin{aligned} L &= \int_1^3 \sqrt{1+(1+\ln x)^2}\,dx \approx \frac{\frac{1}{5}}{3}\left(f(1)+4f(\tfrac{6}{5})+2f(\tfrac{7}{5})+\cdots+4f(\tfrac{14}{5})+f(3)\right) \\ &\approx \tfrac{1}{15}(58.044271) \approx 3.869618 \end{aligned}$$

Section 9.2

2. The integral that gives the area of the surface obtained by rotating $y = \sin^2 x$, $0 \le x \le \frac{\pi}{2}$ about the x-axis is:

$$S = \int_0^{\pi/2} 2\pi\sin^2 x\sqrt{1+4\sin^2\cos^2 x}\,dx$$

20. We use Simpson's Rule with $n = 10$ to approximate the area of the surface obtained by rotating $y = \sqrt{1+e^x}$, $0 \le x \le 1$, about the x-axis. Note that $\frac{dy}{dx} = \frac{1}{2}(1+e^x)^{-1\ 2}e^x$. Thus, the area is:

$$\begin{aligned} S &= \int_0^1 2\pi\sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\,dx = \pi\int_0^1 2\sqrt{1+e^x}\sqrt{\frac{e^{2x}+4e^x+4}{4(1+e^x)}}\,dx \\ &= \pi\int_0^1 2\sqrt{1+e^x}\sqrt{\frac{(e^x+2)^2}{4(1+e^x)}}\,dx = \pi\int_0^1 (e^x+2)\,dx \\ &\approx \pi\frac{\frac{1}{10}}{3}\left((e^0+2)+4(e^{0.1}+2)+2(e^{0.2}+2)+\cdots+(e^1+2)\right) \\ &\approx \frac{\pi}{30}(111.54848) \approx 11.68133 \end{aligned}$$

We can check that we applied Simpson's Rule correctly by evaluating $\pi \int_0^1 (e^x + 2)\, dx$ directly.

Section 9.3

20. We have a mass of 25 at $x = -2$, a mass of 20 at $x = 3$, and a mass of 10 at $x = 7$. By the formula on page 600, the moment is:

$$M = (25)(-2) + (20)(3) + (10)(7) = 80.$$

Since the total mass is $m = 55$, the center of mass is:

$$\bar{x} = \frac{M}{m} = \frac{80}{55} = \frac{16}{11}.$$

32. Since the figure is symmetric left and right, $M_y = 0$ and $\bar{x} = 0$. To calculate M_x, we can add the moments (about the x-axis) of the semicircle and the square. The moment for the square will be negative, however, since the square is below the x-axis. Recalling that the density ρ is 5, we have:

$$M_x = 5\int_{-1}^{1} \tfrac{1}{2}(1-x^2)\, dx - 5\int_{-1}^{1} \tfrac{1}{2}(2)^2\, dx = \tfrac{5}{2}\left[x - \tfrac{1}{3}x^3\right]_{-1}^{1} - 5\left[2x\right]_{-1}^{1} = \tfrac{10}{3} - 20 = -\tfrac{50}{3}$$

Since the total mass is $5(4 + \frac{\pi}{2})$, we find:

$$\bar{y} = \frac{-\frac{50}{3}}{5(4+\frac{\pi}{2})} = \frac{-20}{3(\pi+8)} \approx -0.598$$

Therefore, the center of mass is $(\bar{x}, \bar{y}) = \left(0, \frac{-20}{3(\pi+8)}\right) \approx (0, -0.598)$.

38. We can partition the given region into three rectangles: one with mass 6 centered at $\left(2, \frac{3}{2}\right)$, one with mass 2 centered at $\left(0, \frac{1}{2}\right)$, and one with mass 2 centered at $\left(-\frac{3}{2}, 1\right)$. Thus, $M_y = 6 \cdot 2 + 2 \cdot 0 + 2 \cdot \frac{-3}{2} = 9$ and $M_x = 6 \cdot \frac{3}{2} + 2 \cdot \frac{1}{2} + 2 \cdot 1 = 12$. The total mass is 10, so the centroid is $(\bar{x}, \bar{y}) = \left(\frac{9}{10}, \frac{6}{5}\right)$.

Section 9.4

2. By the Net Change Theorem (see page 354), $R(5000) - R(1000) = \int_{1000}^{5000} R'(t)\, dt$. Thus, we have:

$$R(5000) = 12400 + \int_{1000}^{5000} (12 - 0.0004x)\, dx = 12400 + \left[12x - 0.0002x^2\right]_{1000}^{5000} = 55600.$$

So the revenue from the sale of the first 5000 units is \$55,600.

6. The producer surplus for the supply function $p_s(x) = 3 + 0.01x^2$ at the sales level $X = 10$ is given by:

$$\int_0^X [P - p_s(x)]\ dx = \int_0^{10} \left[4 - (3 + 0.01x^2)\right]\ dx = \int_0^{10} \left[1 - 0.01x^2\right]\ dx = \frac{20}{3}.$$

So the producer surplus is \$6.67. On a graph, the producer surplus is given by the area below $y = 4$ and above $y = 3 + 0.01x^2$, between $x = 0$ and $x = 10$.

Section 9.5

8. (a) The exponential density function (see page 613) for the lifetime of the bulbs is:

$$f(x) = \begin{cases} 0 & \text{if } t < 0 \\ 0.001e^{-0.001t} & \text{if } t \geq 0 \end{cases}$$

Therefore, the probability that a bulb fails within the first 200 hours is:

$$P(T < 200) = \int_0^{200} 0.001e^{-0.001t}\ dt = -e^{-0.001t}\Big|_0^{200} = -e^{-0.2} + e^0 \approx 0.1813$$

The probability that the bulb does not fail in the first 800 hours is:

$$P(T > 800) = \int_{800}^{\infty} 0.001e^{-0.001t}\ dt = -e^{-0.001t}\Big|_{800}^{\infty} = e^{-0.8} \approx 0.4493$$

(b) To find the median lifetime of the bulbs, we want to find T such that $0.5 = \int_0^T f(t)\ dt$. That is:

$$\begin{aligned} 0.5 &= \int_0^T 0.001e^{-0.001t}\ dt \\ 0.5 &= -e^{-0.001T} + 1 \\ -0.5 &= -e^{-0.001T} \\ \ln\tfrac{1}{2} &= -\tfrac{T}{1000} \\ 1000\ln 2 &= T \\ T &\approx 693 \text{ hours} \end{aligned}$$

12. Recall that the probability density function of a normal distribution with mean μ and standard deviation σ (see page 616) is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

(a) Assume the mean $\mu = 500$ g and the standard deviation $\sigma = 12$ g. Then the probability that the machine produces a box with less than 480 g of cereal is

$$P(X < 480) = \int_{-\infty}^{480} f(x)\,dx \approx 0.04779$$

or about 4.8%. (Use a computer or calculator to evaluate the integral.)

(b) We want to find a mean μ such that $P(X < 500) = .95$. Using a calculator to evaluate $\int_{-\infty}^{500} f(x)\,dx$ for various values of μ, we find that if $\mu = 519.73$, then $P(x < 500) = 0.05007$. Or, if $\mu = 519.74$, then $P(x < 500) = 0.04998$. Therefore, we should choose μ at least 519.74 g so that less than 5% of the boxes contain less than 500 g of cereal.