Math 114, Section 10.3, Problem 46

(a) Suppose the resisting force is proportional to the velocity, so $m\frac{dv}{dt} = -kv$, with k a positive constant. Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s.

Separate variables and integrate to obtain $\displaystyle\int \frac{dv}{v} = -\int \frac{k}{m}dt$, so $\ln|v| = -\frac{k}{m}t + C$.

Since $v(0) = v_0$, $C = \ln|v_0|$. Therefore, $\ln\left|\frac{v}{v_0}\right| = -\frac{k}{m}t$, so $v(t) = \pm v_0 e^{-kt/m}$. The sign is $+$ when $t = 0$, so the sign is $+$ for all t since we can assume v is continuous.

Then $v(t) = v_0 e^{-kt/m}$, and $\frac{ds}{dt} = v(t)$, so we integrate again to obtain $s(t) = -\frac{mv_0}{k}e^{-kt/m} + C'$. From $s(0) = s_0$, we find $C' = s_0 + \frac{mv_0}{k}$. Thus,

$$s(t) = s_0 + \frac{mv_0}{k}\left(1 - e^{-kt/m}\right).$$

The distance traveled from time 0 to time t is $s(t) - s_0$, so total distance traveled is

$$\lim_{t\to\infty}(s(t) - s_0) = \frac{mv_0}{k}.$$

(b) Now suppose the resisting force is proportional to the square of the velocity, so $m\frac{dv}{dt} = -kv^2$, with $k > 0$.

Separating variables and integrating, we obtain $\displaystyle\int \frac{dv}{v^2} = -\int \frac{k}{m}dt$, so $\frac{-1}{v} = -\frac{kt}{m} + C$. Since $v(0) = v_0$, we find $C = \frac{-1}{v_0}$, so $\frac{1}{v} = \frac{kt}{m} + \frac{1}{v_0}$. Thus,

$$v(t) = \frac{1}{kt/m + 1/v_0} = \frac{mv_0}{kv_0t + m}.$$

Again, $\frac{ds}{dt} = v(t)$, so we have

$$s(t) = \frac{m}{k}\int \frac{kv_0dt}{kv_0t + m} = \frac{m}{k}\ln|kv_0t + m| + C'.$$

Since $s(0) = s_0$, we find $C' = s_0 - \frac{m}{k}\ln m$. Thus,

$$s(t) = s_0 + \frac{m}{k}(\ln|kv_0t + m| - \ln m) = s_0 + \frac{m}{k}\ln\left|\frac{kv_0t + m}{m}\right| = s_0 + \frac{m}{k}\ln\left(1 + \frac{kv_0}{m}t\right).$$

The distance traveled from time 0 to time t is $s(t) - s_0$, so total distance traveled is

$$\lim_{t\to\infty}(s(t) - s_0) = \frac{m}{k}\left(1 + \frac{kv_0}{m}t\right) = \infty.$$

Thus, the object travels arbitrarily far, given enough time.

Note also that $\lim_{t\to\infty} v(t) = 0$, so the velocity gets arbitrarily close to zero for large enough t.