Math 114, Section 10.4, Problem 17(c)

We will solve explicitly the differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right)$.

First, we simplify the parentheses on the right side to obtain $\frac{dP}{dt} = \frac{k}{K}(K-P)(P-m)$.

Separating variables, we have

$$\int \frac{dP}{(K-P)(P-m)} = \int \frac{k}{K}\,dt. \tag{1}$$

To integrate the left side of (1), we use partial fractions:

$$\frac{1}{(K-P)(P-M)} = \frac{A}{K-P} + \frac{B}{P-M} \quad \text{where} \quad A = B = \frac{1}{K-m}.$$

Thus, the left side of (1) becomes:

$$\begin{aligned} \int \frac{dP}{(K-P)(P-m)} &= \frac{1}{K-m}\int\left(\frac{1}{K-P} + \frac{1}{P-m}\right)dP \\ &= \frac{1}{K-m}\left(-\ln(K-P) + \ln(P-m)\right) \\ &= \frac{1}{K-m}\ln\frac{P-m}{K-P}. \end{aligned} \tag{2}$$

The right side of (1) is:

$$\int \frac{k}{K}\,dt = \frac{k}{K}t + C. \tag{3}$$

Equating (3) and (2), we have

$$\frac{1}{K-m}\ln\frac{P-m}{K-P} = \frac{k}{K} + C. \tag{4}$$

With the initial condition $P(0) = P_0$, we find $C = \frac{1}{K-m}\ln\frac{P_0-m}{K-P_0}$.

Substituting the above expression for C in (4), it is a matter of algebraic manipulation to solve for P, obtaining:

$$P(t) = \frac{m(K-P_0) + K(P_0-m)e^{(K-m)kt/K}}{K-P_0+(P_0-m)e^{(K-m)kt/K}}.$$