Math 114, Section 13.3, Problem 49

We will use a scalar projection to find the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$.

Strategy: Let $\mathbf{n}$ be a vector in the plane orthogonal to the line. Let $\mathbf{v}$ be a vector from any point on the line to the point P_1. The distance from P_1 to the line is given by the length of the projection of $\mathbf{v}$ onto $\mathbf{n}$.

The line $ax + by + c = 0$ has slope $\frac{-a}{b}$, so an orthogonal line has slope $\frac{b}{a}$. A vector in the direction of a line with slope $\frac{b}{a}$ is $\mathbf{n} = \langle a, b \rangle$.

The point $(x, y) = (0, -\frac{c}{b})$ is on the line $ax + by + c = 0$. Thus, a vector from the line to P_1 is $\mathbf{v} = \langle x_1, y + \frac{c}{b} \rangle$.

Therefore, the distance from the point to the line is

$$|\text{comp}_{\mathbf{n}} \mathbf{v}| = \frac{|\langle a, b \rangle \cdot \langle x_1, y_1 + \frac{c}{b} \rangle|}{|\langle a, b \rangle|} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$