Quiz 4

Name: ____________________________

Math 114 - Calculus II
October 8, 2008

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Let $\mathbf{a} = \langle 0, 0, 1\rangle$, $\mathbf{b} = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$, and $\mathbf{c} = \langle \cos\theta, \sin\theta, 0\rangle$. For what value of θ (between 0 and π) is $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ maximized? For this value of θ, what is the shape of the parallelpiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$?
   SOLUTION: First note that it only makes sense to compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Using the determinant mnemonic for the cross product, we have

   $$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)\mathbf{k}.$$

   Next taking the dot product of this with $\mathbf{a}$ gives

   $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta).$$

   We want to maximize this function of θ, so we differentiate it and set the derivative equal to zero:

   $$0 = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta).$$

   The only solution between 0 and π is $\theta = \pi/4$ or $45°$.

   For $\theta = 45°$, the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are unit vectors which are pairwise orthogonal, so the parallelpiped determined by them is a cube.

2. Find the plane in $\mathbb{R}^3$ that contains the point $(3, 2, 1)$ and the line $\mathbf{r}(t) = \langle 1 + t, 2 + 2t, 3 + 3t\rangle$.

   SOLUTION: First we find two points on the line: $\mathbf{r}(-1) = (0, 0, 0)$ and $\mathbf{r}(0) = (1, 2, 3)$. These two points together with the point $(3, 2, 1)$ mean that the vectors

   $$\mathbf{u} = (3, 2, 1) - (0, 0, 0) = (3, 2, 1) \quad \text{and} \quad \mathbf{v} = (1, 2, 3) - (0, 0, 0) = (1, 2, 3)$$

   lie in the plane; hence their cross product is orthogonal (or normal) to the plane and can thus be taken to be the normal vector to the plane. That is, we can set

   $$\mathbf{n} = \mathbf{u} \times \mathbf{v} = (4, -8, 4)$$

   (where we compute the cross product as in the first problem). With the normal vector and a point, say, $(0, 0, 0)$ on the plane, we can write down the equation of the plane:

   $$0 = 4(x - 0) - 8(y - 0) + 4(z - 0), \quad \text{or} \quad x - 2y + z = 0.$$

Quiz 4 Name: ______________________

Math 114 - Calculus II

October 10, 2008

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Let a, b, c be real numbers, and consider the line $\mathbf{r}(t) = \langle 11 + at, 22 + bt, 33 + ct\rangle$ in $\mathbb{R}^3$. For what values of A is the vector $\mathbf{u} = \langle A, ab + b, ac + c\rangle$ parallel to $\mathbf{r}(t)$? For what values is $\mathbf{u}$ perpendicular to $\mathbf{r}(t)$?

   SOLUTION: Write $\mathbf{r}(t)$ as $\mathbf{r}_0 + t\mathbf{v}$ where $\mathbf{r}_0 = (11, 22, 33)$ and $\mathbf{v} = (a, b, c)$. Then $\mathbf{u}$ is parallel to $\mathbf{r}(t)$ when $\mathbf{u}$ is parallel to $\mathbf{v}$, which happens precisely when $\mathbf{u} = k\mathbf{v}$ for some scalar k. This vector equation gives three scalar equations:

   $$A = ka, \quad ab + b = kb, \quad \text{and} \quad ac + c = kc.$$

   From either of the last two equations, we see that $k = a + 1$, so from the first equation, we conclude $A = a(a + 1)$. To find the value of A such that $\mathbf{u}$ is perpendicular to $\mathbf{r}(t)$ (or, equivalently, to $\mathbf{v}$), we find A such that

   $$0 = \mathbf{v} \cdot \mathbf{u} = aA + b(ab + b) + c(ac + c) = aA + (a + 1)(b^2 + c^2).$$

   Solving for A, we obtain $A = -\dfrac{a+1}{a}(b^2 + c^2)$.

2. Find the plane in $\mathbb{R}^3$ that contains the points $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$.

   SOLUTION: Given these three points on the plane, we can find two vectors on the plane, say,

   $$\mathbf{u} = (2, 3, 1) - (1, 2, 3) = (1, 1, -2), \quad \text{and} \quad \mathbf{v} = (3, 1, 2) - (1, 2, 3,) = (2, -1, -1).$$

   The cross product of these two vectors will be orthogonal to $\mathbf{u}$ and $\mathbf{v}$; hence, since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, the cross product will be normal to the plane. Hence we can set the normal vector $\mathbf{n}$ to the plane to be

   $$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} = (-3, -3, -3).$$

   With the normal vector to the plane and a point, say, $(1, 2, 3)$, on the plane, we obtain the equation of the plane:

   $$0 = -3(x - 1) - 3(y - 2) - 3(z - 3), \quad \text{or} \quad x + y + z = 6.$$

   Alternatively, since we know that three non-colinear points determine a unique plane, and since $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$ are three non-colinear that satisfy $x + y + z = 6$, we could have concluded with ease that the plane is $x + y + z = 6$.