Quiz 6

Name: ______________________

Math 114 - Calculus II
October 22, 2008

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Evaluate $\int_0^1 \mathbf{r}'(t)dt$ and $\int_0^1 |\mathbf{r}'(t)|dt$ where $\mathbf{r}(t) = \langle 3\cos(\pi e^t), 4\pi e^t, 3\sin(\pi e^t)\rangle$.

   SOLUTION: By the fundamental theorem of calculus,

$$\begin{aligned}\int_0^1 \mathbf{r}'(t)dt &= \mathbf{r}(1) - \mathbf{r}(0) = (3\cos(\pi e), 4\pi e, 3\sin(\pi e)) - (3\cos\pi, 4\pi, 3\sin\pi) \\ &= (3\cos(\pi e) + 3, 4\pi e - 4\pi, 3\sin(\pi e)).\end{aligned}$$

   For the second, note that $|\mathbf{r}'(t)| \neq \frac{d}{dt}|\mathbf{r}(t)|$, so the answer is **not** $|\mathbf{r}(1)| - |\mathbf{r}(0)|$; you have to compute directly:

$$\begin{aligned}|\mathbf{r}'(t)| &= \sqrt{(-3\pi e^t \sin(\pi e^t))^2 + (4\pi e^t)^2 + (3\pi e^t \cos(\pi e^t))^2} \\ &= \pi e^t \sqrt{9\sin^2(\pi e^t) + 16 + 9\cos^2(\pi e^t)} = \pi e^t \sqrt{9+16} = 5\pi e^t,\end{aligned}$$

   so

$$\int_0^1 |\mathbf{r}'(t)|dt = \int_0^1 5\pi e^t dt = 5\pi(e-1).$$

2. Let v denote the speed of a particle moving with velocity $\mathbf{v}$. Then $v^2 = |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$. Find $v' = \frac{dv}{dt}$ if the acceleration $\mathbf{a} = \mathbf{v}'$ is $\langle 1, 2, 3\rangle$ and the velocity $\mathbf{v}$ is $3\mathbf{i} + 4\mathbf{k}$.

   SOLUTION: From the hint, $v^2 = \mathbf{v}\cdot\mathbf{v}$, so we can implicitly differentiate with respect to t to get $2v\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})$. By the product rule,

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 2\mathbf{a}\cdot\mathbf{v}$$

   because $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ by definition and $\mathbf{v}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{v}$. Solving, we have the expression $\frac{dv}{dt} = \frac{\mathbf{a}\cdot\mathbf{v}}{v}$. If $\mathbf{a} = (1,2,3)$ and $\mathbf{v} = (3,0,4)$, then $v = |\mathbf{v}| = \sqrt{3^2+4^2} = 5$, and hence

$$\frac{dv}{dt} = \frac{1(3) + 2(0) + 3(4)}{5} = 3.$$

   Alternatively, if you did not think to implicitly differentiate, you could solve for v then take the derivative: Solving gives $v = \sqrt{\mathbf{v}\cdot\mathbf{v}}$, so

$$\frac{dv}{dt} = \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})^{-1/2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}).$$

   Using the hint that $v^2 = \mathbf{v}\cdot\mathbf{v}$, this gives the same result as above.

Quiz 6 Name: ____________________

Math 114 - Calculus II

October 24, 2008

Note: *In order to receive full credit, you must show work that justifies your answer.*

Recall that the force applied to an object is equal to the objects' mass times its acceleration.

1. Suppose a particle of unit mass is moving along the y-axis (in the positive y-direction) with a speed of 5 m/s. At time $t = 0$, the particle reaches the origin. Just as the particle enters the upper-half plane, a mighty being (e.g., a scientist!) applies a force $\mathbf{F}(t) = \langle t, e^t - 1\rangle$ to the object. Calculate the particle's trajectory $\mathbf{r}(t)$ for $t \geq 0$.

   SOLUTION SKETCH: It is given that $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}'(0) = (0, 5)$. It is also given that the mass m is 1, so Newton's law gives $\mathbf{F}(t) = m\mathbf{r}''(t) = \mathbf{r}''(t)$. Integrating once, we get

   $$\mathbf{r}'(t) = \left(\frac{1}{2}t^2, e^t - t\right) + (C, D).$$

   Evaluating this at $t = 0$ and using the initial condition $\mathbf{r}'(0) = (0, 5)$, we find $(C, D) = (0, 4)$, so $\mathbf{r}'(t) = \left(\frac{1}{2}t^2, e^t - t + 4\right)$. Similarly, we have

   $$\mathbf{r}(t) = \left(\frac{1}{6}t^3, e^t - \frac{1}{2}t^2 + 4t - 1\right)$$

   by integrating and using the initial condition $\mathbf{r}(0) = \mathbf{0} = (0, 0)$.

2. Find $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v})$ where $\mathbf{u} = \langle e^t, e^t, e^t\rangle$ and $\mathbf{v} = \langle t^3, \ln t, \ln t\rangle$.

   SOLUTION: You could have either taken the dot product then differentiated or differentiated first using the product rule. To see an instance of the product rule, see Wednesday's quiz. I will take the derivative the first way (though it sometimes saves time to use the product rule!):

   $$\begin{aligned} \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) &= \frac{d}{dt}(e^t t^3 + e^t \ln t + e^t \ln t) \\ &= \frac{d}{dt}\left(e^t(t^3 + 2\ln t)\right) \\ &= e^t(t^3 + 2\ln t) + e^t(3t^2 + \frac{2}{t}) \\ &= e^t(t^3 + 3t^2 + 2\ln t + \frac{2}{t}). \end{aligned}$$

   NOTE: The result is a scalar, not a vector. Dot products output scalars. Dot products give you scalars. The dot product of two vectors is a scalar. Got it? :-)