Quiz 9

Name: ______________________________

Math 114 - Calculus II

November 12, 2008

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Let f be the function given by $f(x, y) = y^2 - \cos x$.

   (a) Compute the gradient of f at the point $(0, 1)$.

   SOLUTION: $\nabla f(0,1) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{(0,1)} = (\sin x, 2y)|_{(0,1)} = (0, 2)$.

   (b) Compute the directional derivative of f at $(0, 1)$ in the direction of $\mathbf{u} = (1, 1)$.

   SOLUTION: First normalize $\mathbf{u}$ to get $\frac{1}{|(1,1)|}(1,1) = \frac{1}{\sqrt{2}}(1,1)$. Next, compute

   $$D_{\mathbf{u}} f(0,1) = \nabla f(0,1) \cdot \left(\frac{1}{\sqrt{2}}(1,1)\right) = \sqrt{2}.$$

   (c) In which direction is f increasing the fastest?

   SOLUTION: f increases fastest in the direction of the gradient.

2. Find three positive numbers p, q, and r whose sum is 1 and whose product $P = pqr$ is maximized.

   SOLUTION: Since $p + q + r = 1$, we can write P as a function of two variables, say,

   $$P = P(p, q) = pq(1 - p - q).$$

   Then to maximize P, we find critical points, i.e., points (x, y) such that ∇P is undefined or

   $$0 = \nabla P = \left(\frac{\partial P}{\partial p}, \frac{\partial P}{\partial q}\right) = (q - 2pq - q^2, p - 2pq - p^2) = (q(1 - 2p - q), p(1 - 2q - p)).$$

   The gradient is always defined, so solving $\nabla P = 0$, and recalling that p and q are positive (and, therefore, nonzero), we find $p = q = 1/3$. Hence $p = 1/3$, $q = 1/3$, and $r = 1/3$ sum to 1 and maximize P.

Quiz 9

Name: ______________________

Math 114 - Calculus II
November 14, 2008

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Let f be the function given by $f(x,y) = x - e^{y^2}$.

   (a) Compute the gradient of f at $(5,0)$.
   SOLUTION: $\nabla f(5,0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{(5,0)} = (1, -2ye^{y^2})\Big|_{(5,0)} = (1,0)$.

   (b) Compute the directional derivative of f at $(5,0)$ in the direction of $\mathbf{u} = (-1,2)$.
   SOLUTION: First normalize $\mathbf{u}$ to get $\frac{1}{\sqrt{5}}(-1,2)$. Then

   $$D_{\mathbf{u}} f(5,0) = \nabla f(5,0) \cdot \left(\frac{1}{\sqrt{5}}(-1,2)\right) = -\frac{1}{\sqrt{5}}.$$

   (c) In which direction is f increasing the fastest?
   SOLUTION: In the direction of the gradient!

2. Write a formula for the distance D from the origin to a point (a,s,h) on the plane given by the equation $x + 4 = y$. Find the minimum distance. [Hint: Minimize D^2 first.]

   SOLUTION: A point (a,s,h) on the plane $x+4=y$ satisfies $s = a+4$, so the square of the distance D from the $(0,0,0)$ to such a point (a,s,h) is

   $$D^2 = a^2 + s^2 + h^2 = a^2 + (a+4)^2 + h^2 = 2a^2 + 8a + 16 + h^2.$$

   Since D is nonnegative, and since the function $f(t) = t^2$ is increasing for nonnegative t, D is minimized precisely when $D^2 = f(D)$ is minimized. Hence it suffices to minimize D^2. To do this, find its critical points, i.e., the points (a,h) where

   $$\nabla(D^2) = \left(\frac{\partial(D^2)}{\partial a}, \frac{\partial(D^2)}{\partial h}\right) = (4a+8, 2h)$$

   is $\mathbf{0}$ or undefined. Clearly it is always defined so the only critical point (a,h) is the point $(-2,0)$. Since $s = a+4$, $s = 2$. So the mininum distance is
   $D(-2,2,0) = \sqrt{4+4+0} = \sqrt{8} = 2\sqrt{2}$.