Section 5.2, Problem 16

The point $x = 0$ is a regular singular point of the differential equation

$$2xy'' + 5y' + xy = 0.$$

We use the method of Frobenius to obtain two linearly independent series solutions about $x = 0$. Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Then $y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$.
Note that when finding the derivatives, the indices remain $y = 0$ to ∞ since we cannot assume that r is an integer and thus the power series might not have a constant term (or any low order terms, for that matter) that becomes zero upon differentiation.
Substituting into the differential equation, we have:

$$\begin{aligned}
2xy'' + 5y' + xy &= 2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 5\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + x\sum_{n=0}^{\infty}c_n x^{n+r} \\
&= 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + 5\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}c_n x^{n+r+1} \\
&= 2\sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^{k+r-1} + 5\sum_{k=0}^{\infty}(k+r)c_k x^{k+r-1} + \sum_{k=2}^{\infty}c_{k-2} x^{k+r-1} \\
&= 2r(r-1)c_0 x^{r-1} + 2(r+1)rc_1 x^r + 5rc_0 x^{r-1} + 5(r+1)c_1 x^r \\
&\qquad + \sum_{k=2}^{\infty}\left[2(k+r)(k+r-1)c_k + 5(k+r)c_n + c_{n-2}\right]x^{k+r-1} \\
&= r(2r+3)c_0 x^{r-1} + (2r^2+7r+5)c_1 x^r \\
&\qquad + \sum_{k=2}^{\infty}\left[(n+r)(2n+2r+3)c_n + c_{n-2}\right]x^{k+r-1} = 0
\end{aligned}$$

From the terms in front of the sum, we have $r(2r+3)c_0 = 0$ and $(2r^2+7r+5)c_1 = 0$. However, we can assume that $c_0 \neq 0$, since if $c_0 = 0$ then the first term of our series is zero and we have the wrong r. (That is, we can decrease r by 1 and begin the series at the next term.) Thus, the indicial equation is $r(2r+3) = 0$, so the indicial roots are $r = 0$ and $r = -\frac{3}{2}$. Since neither of these satisfy the equation for c_1, it must be that $c_1 = 0$.
The root $r = 0$ gives us one solution: the recursion is $c_k = \dfrac{-c_{k-2}}{k(2k+3)}$, for $k = 2, 3, 4, \ldots$. This gives $c_2 = -\frac{1}{14}c_0$, $c_3 = 0$, $c_4 = \frac{1}{616}c_0$, etc.
The root $r = -\frac{3}{2}$ gives us another solution: the recursion is $c_k = \dfrac{-c_{k-2}}{k(2k-3)}$, for $k = 2, 3, 4, \ldots$. This gives $c_2 = -\frac{1}{2}c_0$, $c_3 = 0$, $c_4 = \frac{1}{40}c_0$, etc.
Therefore, the general solution is:

$$y = Ax^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \cdots\right) + B\left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \cdots\right).$$