Homework 3, Selected Solutions

Section 7.7, problem 17

We are given the basis $u_1 = 1$, $u_2 = x$, and $u_3 = x^2$. To transform this basis into an orthogonal basis, we use the Gram-Schmidt Process (Theorem 7.6, page 344).

Note: The equations of Theorem 7.6 make the Gram-Schmidt Process look confusing, but it's much easier to understand geometrically. Ask in office hours or recitation if you want the geometric picture.

The first vector of our new basis is $v_1 = u_1 = 1$.

The second vector is $v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$, where instead of the regular dot product for vectors, we have the "inner product" $(p, q) = \int_{-1}^{1} p(x)q(x)\,dx$. That is,

$$(u_2, v_1) = \int_{-1}^{1} x\,dx = 0 \quad \text{and} \quad (v_1, v_1) = \int_{-1}^{1} 1\,dx = 2.$$

Thus, $v_2 = u_2 - 0 = u_2 = x$.

The third vector is $v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$.

$$(u_3, v_1) = \int_{-1}^{1} x^2\,dx = \frac{2}{3}, \quad (u_3, v_2) = \int_{-1}^{1} x^3\,dx = 0 \quad \text{and} \quad (v_2, v_2) = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

Thus, $v_3 = x^2 - \frac{2/3}{2} 1 - \frac{0}{2/3} x = x^2 - \frac{1}{3}$.

Therefore, an orthogonal basis is $v_1 = 1$, $v_2 = x$, and $v_3 = x^2 - \frac{1}{3}$. The answer in the back of the book is also an orthonormal basis (v_3 is different by a factor of $\frac{3}{2}$), as you can check by computing the inner products (v_1, v_2), (v_1, v_3), and (v_2, v_3).

This question does not ask for an ortho*normal* basis; if it did, we would have to normalize each basis element by dividing by its "length." In this context, the "length" of a function $p(x)$ is defined

$$|||p(x)||| = (p, p)^{1/2} = \left(\int_{-1}^{1} p^2(x)\,dx \right)^{1/2}.$$

Section 8.10, problem 16

We want to construct an orthogonal matrix from the eigenvectors of the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

First, find the eigenvalues:

$$\det(\mathbf{A} - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda^3 - \lambda^2 - 3\lambda - 1).$$

We might have trouble factoring the polynomial above, but the Rational Root Theorem says that the only possible rational (in fact, integer) roots are 1 and -1. We find that $\lambda = -1$ is a root. Thus,

$$\det(\mathbf{A} - \lambda I) = -(\lambda + 1)(\lambda^2 - 2\lambda - 1).$$

Using the quadratic formula to find the roots of the quadratic part, we find the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1 - \sqrt{2}$, and $\lambda_3 = 1 + \sqrt{2}$.

Next, find the eigenvectors:

For $\lambda_1 = -1$, we find that an eigenvector is $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 1 - \sqrt{2}$, row-reduction is a bit messy due to the square roots, but

$$[\mathbf{A} - \lambda_2 \mathbf{I}] = \left[\begin{array}{ccc|c} \sqrt{2}-1 & 1 & 1 & 0 \\ 1 & \sqrt{2} & 1 & 0 \\ 1 & 1 & \sqrt{2}-1 & 0 \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so an eigenvector is $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$.

Similarly, for $\lambda_3 = 1 + \sqrt{2}$, an eigenvector is $\mathbf{u}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$.

These three eigenvectors are mutually orthogonal, but they are not unit vectors. We divide each eigenvector by its length to obtain unit eigenvectors: $\mathbf{v}_1 = \frac{\mathbf{u}_1}{\sqrt{2}}$, $\mathbf{v}_2 = \frac{\mathbf{u}_2}{2}$, and $\mathbf{v}_3 = \frac{\mathbf{u}_3}{2}$.

Finally, we obtain the desired orthogonal matrix, whose columns are the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$