Homework 5, Selected Solutions

Section 3.4, problem 22
We will solve the differential equation: $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$ using the "annihilator" method. Written in terms of differential operators, the equation is:

$$\begin{aligned}(D^3 - 2D^2 - 4D + 8)y &= 6xe^{2x} \\ (D-2)^2(D+2)y &= 6xe^{2x}\end{aligned}$$

and the complementary solution is evidently $y_c = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x}$.
To find the annihilator of the right side, observe that $D-2$ annihilates e^{2x}, and $(D-2)^2$ annihilates xe^{2x}. Multiplying the equation by the annihilator (in blue), we have:

$$(D-2)^2(D-2)^2(D+2)y = (D-2)^2 6xe^{2x} = 0.$$

The annihilator tells us that the particular solution will have two terms of the form e^{2x}. We already have two such terms in the complementary solution, so the particular solution is

$$y_p = (Ax^2 + Bx^3)e^{2x}.$$

To find the constants A and B, we differentiate y_p. Simplifying as we go makes this somewhat easier:

$$\begin{aligned}y_p' &= (2Ax + 2Ax^2 + 3Bx^2 + 2Bx^3)e^{2x} \\ y_p'' &= (2A + 8Ax + 6Bx + 4Ax^2 + 12Bx^2 + 4Bx^3)e^{2x} \\ y_p''' &= (12A + 6B + 24Ax + 36Bx + 8Ax^2 + 36Bx^2 + 8Bx^3)e^{2x}\end{aligned}$$

Substituting these derivatives of y_p into the differential equation and simplifying, we have:

$$y''' - 2y'' - 4y' + 8y = (8A + 6B + 24Bx)e^{2x} = 6xe^{2x}.$$

Thus, $B = \frac{1}{4}$ and $A = -\frac{3}{16}$.
The general solution is therefore

$$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + \left(\frac{1}{4}x^3 - \frac{3}{16}x^2\right)e^{2x}.$$