Quiz 2 Name: ______________________

Math 240 - Calculus III

February 3, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \quad \textit{clear first column}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad \textit{clear second column}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \textit{obtain 1 in lower right}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \textit{done!}$$

Therefore, $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

2. Find all eigenvalues and at least one eigenvector of the matrix $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

Solution: The eigenvalues are the numbers λ that satisfy $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$.

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

So $\lambda = -2$ and $\lambda = 5$ are the eigenvalues.

For $\lambda = -2$, the eigenvector $\mathbf{x}$ satisfies $(\mathbf{B} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$, and we find that $\mathbf{x}$ has the form $\begin{bmatrix} -t \\ t \end{bmatrix}$.

For $\lambda = 5$, the eigenvector $\mathbf{x}$ satisfies $(\mathbf{B} - 5\mathbf{I})\mathbf{x} = \mathbf{0}$, and $\mathbf{x}$ has the form $\begin{bmatrix} 3t \\ 4t \end{bmatrix}$.

Quiz 2 Name: ______________________________

Math 240 - Calculus III

February 5, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Compute the determinant of $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 4 & 7 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}$.

   Solution: Row-reducing is one way to find the determinant:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 0 & 4 & 7 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 3 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -6 & 1 & 4 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & -6 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$
$$= 24$$

   If you know the "cofactor expansion" method, you can find the answer more quickly. Observing that the fourth column contains three zeros, we have:

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 0 & 4 & 7 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 3 & 0 & 0 \end{vmatrix} = -4\begin{vmatrix} 0 & 4 & 7 \\ 0 & 2 & 2 \\ 1 & 3 & 0 \end{vmatrix} = -4(1)\begin{vmatrix} 4 & 7 \\ 2 & 2 \end{vmatrix} = -4(8-14) = 24.$$

2. Let $\mathbf{B} = \begin{bmatrix} 2 & 5 & 8 \\ 0 & 9 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Find the eigenvalues of $\mathbf{B}^{-1}$.

   Solution: You could find the eigenvalues by first computing $\mathbf{B}^{-1}$, but it's much faster to realize that if λ is an eigenvalue of $\mathbf{B}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{B}^{-1}$. The eigenvalues of $\mathbf{B}$ are $\lambda = 1, 2, 9$. Thus, the eigenvalues of $\mathbf{B}^{-1}$ are $\lambda = 1, \frac{1}{2}, \frac{1}{9}$.