Quiz 4 Name: ____________________________

Math 240 - Calculus III

February 24, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (6 points) Find the general solution of the following system of differential equations:

$$\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^{3t}$$

Solution: The matrix $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$, with corresponding eigenvectors $\mathbf{v}_1 = \langle -1, 1 \rangle$ and $\mathbf{v}_2 = \langle 1, 1 \rangle$, respectively. The complementary solution is then

$$\mathbf{X}_c = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}.$$

We suppose a particular solution has the form $\mathbf{X}_p = \begin{bmatrix} a \\ b \end{bmatrix} e^{3t}$. Substituting $\mathbf{X}_p$ into the differential equation, we find that $a = 1$ and $b = 0$.

The general solution is thus

$$\mathbf{X} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}.$$

2. (2 points each) Find differential operators that "annihilate" the following functions:

   (a) $y(x) = e^{5x}$

   Solution: The annihilator is $D - 5$ since

$$(D-5)y = y' - 5y = 5e^{5x} - 5e^{5x} = 0.$$

   (b) $y(x) = xe^x$

   Solution: The annihilator is $(D-1)^2$ since

$$(D-1)^2 y = (D^2 - 2D + 1)y = y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0.$$

Quiz 4

Name: ____________________

Math 240 - Calculus III
February 26, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (6 points) Find the general solution for the following differential equation:

$$y'' - 5y' + 6y = 2x + e^{2x}$$

Solution: The auxiliary equation of the left side is $(D-2)(D-3)$. For the right side, you can multiply the annihilators of $2x$ and e^{2x} to obtain $D^2(D-2)$. Thus, written in terms of differential operators, the equation becomes

$$D^2(D-2)^2(D-3)y = 0.$$

The solution is then

$$y = c_1e^{2x} + c_2e^{3x} + A + Bx + Cxe^{2x}$$

where c_1 and c_2 are coefficients of the complementary solution, and A, B, C are constants we must find for the particular solution.

Substituting $y_p = A + Bx + Cxe^{2x}$ into the differential equation and collecting like terms, we find that $A = \frac{5}{18}$, $B = \frac{1}{3}$, and $C = -1$.

The solution is $y = c_1e^{2x} + c_2e^{3x} + \frac{5}{18} + \frac{1}{3}x - xe^{2x}$.

2. (4 points) "Uncouple" the following system of equations. That is, make a change of variables $\mathbf{X} = \mathbf{PY}$ for an appropriate matrix $\mathbf{P}$ such that the resulting system $\mathbf{Y}' = \mathbf{DY}$ contains a diagonal matrix $\mathbf{D}$.

$$\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{X}$$

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$. The eigenvalues of $\mathbf{A}$ are 1 and 2, and we find the eigenvalues to obtain $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, with

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, the system is "uncoupled" by the change of variable $\mathbf{X} = \mathbf{PY}$, that is

$$\mathbf{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) + 3y_2(t) \\ y_2(t) \end{bmatrix}.$$

The "uncoupled" system is $\mathbf{Y}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{Y}$, with solution $y_1(t) = e^t$ and $y_2(t) = e^{2t}$.