Quiz 5

Name: ______________________________

Math 240 - Calculus III
March 3, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. Solve the following initial value problem for $y(t)$:

$$4t^2y'' + y = 0, \qquad y(1) = 4, \quad y'(1) = 0$$

Solution: We can convert the differential equation into one with constant coefficients by the change of variable $t = e^x$. Then $\frac{dy}{dt} = \frac{dy}{dx}\frac{1}{t}$ and $\frac{d^2y}{dt^2} = \left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\frac{1}{t^2}$, and the equation becomes

$$4\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + y = 4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0.$$

The auxiliary equation is $(2m-1)^2 = 0$, which has a repeated root of $\frac{1}{2}$. The general solution is then $y(x) = c_1e^{x/2} + c_2xe^{x/2}$. But $x = \ln t$, so $y(t) = c_1t^{1/2} + c_2t^{1/2}\ln t$.

The condition $y(1) = 4$ implies $c_1 = 4$, and the condition $y'(1) = 0$ implies $c_2 = -2$. Thus, the solution is

$$y(t) = 4t^{1/2} - 2t^{1/2}\ln t.$$

Note that we could also find the solution using the "book method" of assuming $y = t^m$, from which we obtain $4t^2y'' + y = [4m(m-1)+1]t^m = 0$. We find a repeated root at $t = \frac{1}{2}$ and the solution $y(t) = 4t^{1/2} - 2t^{1/2}\ln t$.

2. A weight of 10 lbs stretches a spring 5 ft. A weight of 8 lbs is suspended from the spring and set in motion. The motion of the system satisfies the differential equation

$$mx'' + \beta x' + kx = 0$$

where m is mass, β is the damping constant, and k is the spring constant. Find β so that the system is *critically damped*.

Solution: First find the constants m and k:

If 10 lbs stretches the spring 5 feet, then $k = \dfrac{10 \text{ lbs}}{5 \text{ ft}} = 2\dfrac{\text{lbs}}{\text{ft}}$.

Since $F = ma$, a weight of 8 lbs has mass $m = \dfrac{8 \text{ lbs}}{32 \text{ ft/sec}^2} = \dfrac{1}{4}$ slug.

The motion of the system then satisfies the differential equation

$$\frac{1}{4}x'' + \beta x' + 2x = 0 \qquad \text{or} \qquad x'' + 4\beta x' + 8x = 0.$$

Roots of the auxiliary equation are $m = \dfrac{-4\beta \pm \sqrt{(4\beta)^2 - 32}}{2}$. The system is critically damped when the auxiliary equation has one repeated root, that is, when $(4\beta)^2 - 32 = 0$, or when $\beta^2 = 2$. We want β to be positive, so $\beta = \sqrt{2}$.

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1. Find the general solution $y(x)$ for the following differential equation:

$$x^2y'' + 3xy' - 4y = 0$$

Solution: We can convert the differential equation into one with constant coefficients by the change of variable $x = e^t$. Then $\frac{dy}{dx} = \frac{dy}{dt}\frac{1}{x}$ and $\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\frac{1}{x^2}$, and the equation becomes

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 3\frac{dy}{dt} - 4y = \frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4y = 0.$$

The auxiliary equation is $m^2 + 2m - 4 = 0$, which has roots $m = -1 \pm \sqrt{5}$. The general solution is then $y(t) = c_1e^{-1+\sqrt{5}t} + c_2e^{-1-\sqrt{5}t}$. But $t = \ln x$, so

$$y(x) = c_1x^{-1+\sqrt{5}} + c_2x^{-1-\sqrt{5}}.$$

Note that we could also find the solution using the "book method" of assuming $y = x^m$, from which we obtain $x^2y'' + 3xy' - 4y = [m(m-1) + 3m - 4]x^m = 0$. We find the roots are $m = -1 \pm \sqrt{5}$ and the solution $y(x) = c_1x^{-1+\sqrt{5}} + c_2x^{-1-\sqrt{5}}$.

2. Find the eigenvalues and corresponding eigenfunctions for the following boundary-value problem:

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y(\pi/4) = 0$$

Solution: Consider the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

If $\lambda < 0$, then let $\alpha^2 = -\lambda$, and the general solution is $y = c_1e^{\alpha x} + c_2e^{-\alpha x}$. The condition $y'(0) = 0$ implies $c_1 = c_2$, so $y = c_1\left(e^{\alpha x} + e^{-\alpha x}\right)$. The condition $y(\pi/4) = 0$ implies $c_1 = 0$, since exponential functions are never zero. Thus, we have only the trivial solution $y = 0$.

If $\lambda = 0$, then $y'' = 0$, so $y = c_1 + c_2x$. Again, the boundary values imply that $c_1 = c_2 = 0$, and we have only the trivial solution $y = 0$.

If $\lambda > 0$, then let $\lambda = \alpha^2$, and the general solution is $y = c_1\cos(\alpha x) + c_2\sin(\alpha x)$. The condition $y'(0) = 0$ implies $c_2 = 0$, so $y = c_1\cos(\alpha x)$. So we want $y(\pi/4) = c_1\cos(\alpha\pi/4) = 0$. This occurs whenever $\alpha\frac{\pi}{4}$ is an odd multiple of $\frac{\pi}{2}$:

$$\alpha\frac{\pi}{4} = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots \qquad \text{or} \qquad \alpha = 4n - 2, \quad n = 1, 2, 3, \ldots$$

Therefore, the eigenvalues are $\lambda = \alpha^2 = (4n-2)^2$, with corresponding eigenfunctions $y(x) = \cos((4n-2)x)$, for $n = 1, 2, 3, \ldots$.