**Quiz 6** Name: ______________________________

Math 240 - Calculus III

March 17, 2009

**Note:** *In order to receive full credit, you must show work that justifies your answer.*

Suppose the position at time $t$ of an object in motion is given by $x(t)$, satisfying

$$x'' - \omega^2 x = -g\sin(\omega t)$$

with initial conditions $x(0) = 0$ and $x'(0) = v_0$. Note that $\omega$ and $g$ are fixed constants.

(a) (7 points) Find the equation of motion $x(t)$.

**Solution**: The complementary solution is $x_c = c_1 e^{\omega t} + c_2 e^{-\omega t}$.

The particular solution has the form $x_p = A\sin(\omega t) + B\cos(\omega t)$. Substituting $x_p$ into the differential equation, we have:

$$x_p'' - \omega^2 x_p = -2\omega^2(A\sin(\omega t) + B\cos(\omega t)) = -g\sin(\omega t).$$

Thus, $B = 0$ and $-2A\omega^2 = -g$, so $A = \frac{g}{2\omega^2}$.

The general solution is then $x_g(t) = c_1 e^{\omega t} + c_2 e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$.

The initial condition $x(0) = 0$ implies $0 = c_1 + c_2$, or $c_1 = -c_2$.

The initial condition $x'(0) = v_0$ implies

$$v_0 = \omega c_1 - \omega c_2 + \frac{g}{2\omega} = -2\omega c_2 + \frac{g}{2\omega}.$$

So $c_2 = \frac{g - 2\omega v_0}{4\omega^2} = -c_1$, and the equation of motion is

$$x(t) = \frac{2\omega v_0 - g}{4\omega^2} e^{\omega t} + \frac{g - 2\omega v_0}{4\omega^2} e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t).$$

(b) (3 points) For what initial velocity $v_0$ will the object exhibit simple harmonic motion?

**Solution**: The object exhibits simple harmonic motion exactly when $c_1 = c_2 = 0$, so $x(t) = \frac{g}{2\omega^2}\sin(\omega t)$.

This occurs when $v_0 = \frac{g}{2\omega}$.

# Quiz 6

Name: ______________________________

Math 240 - Calculus III
March 19, 2009

**Note:** *In order to receive full credit, you must show work that justifies your answer.*

(a) (6 points) Given that $y = \sin x$ is a solution of $y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0$, find the general solution.

**Solution**: The auxiliary equation is $m^4 + 2m^3 + 11m^2 + 2m + 10 = 0$.

Since $y = \sin x$ is a solution, we know that $y = \cos x$ is also a solution, and the auxiliary equation has roots $\pm i$. Thus, the equation contains a factor of $(x^2 + 1)$, and we can factor it as

$$m^4 + 2m^3 + 11m^2 + 2m + 10 = (m^2 + 1)(m^2 + 2m + 10) = 0.$$

The second quadratic factor has roots $m = -1 \pm 3i$. The general solution is thus

$$y(x) = c_1 \sin x + c_2 \cos x + e^{-x}(c_3 \sin(3x) + c_4 \cos(3x)).$$

(b) (4 points) What is the long-term behavior of the solution? Be as specific as possible. (For example, as $x \to \infty$ can you approximate $y(x)$ by a simpler function?)

**Solution**: As $x \to \infty$, $e^{-x} \to 0$. Since $\sin x$ and $\cos x$ are bounded, $e^{-x}(c_3 \sin(3x) + c_4 \cos(3x)) \to 0$.

Thus, for large $x$, $y(x) \approx c_1 \sin x + c_2 \cos x$, so the long-term behavior approaches simple harmonic motion with period $2\pi$.