Quiz 8

Name: ________________________

Math 240 - Calculus III
April 7, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (6 points) Define functions f and g such that

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{xy} \\ \sqrt{x^2+y^2} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3u-3 \\ u^2-5v \end{pmatrix}.$$

If $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ and the rate of change of the outputs is $\begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix} = \begin{pmatrix} 60 \\ 12 \end{pmatrix}$, find $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$, the rate of change of the inputs.

Solution: By the chain rule,

$$\begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix} = \left[D(f\circ g)\begin{pmatrix} u \\ v \end{pmatrix}\right]\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \left[Df\left(g\begin{pmatrix} u \\ v \end{pmatrix}\right)\right]\left[Dg\begin{pmatrix} u \\ v \end{pmatrix}\right]\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}.$$

Computing the determinant matrices:

$$\begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix} = \begin{bmatrix} ye^{xy} & xe^{xy} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2u & -5 \end{bmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}.$$

When $u = v = 5$, $x = 12$ and $y = 0$, so we have

$$\begin{pmatrix} 60 \\ 12 \end{pmatrix} = \begin{bmatrix} 0 & 12 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 10 & -5 \end{bmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 120 & -60 \\ 3 & 0 \end{bmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$$

which we solve to find $\dot{u} = 4$ and $\dot{v} = 7$.

2. (4 points) Find the directional derivative of $f(x,y) = e^{-x}\cos y$ at $(0, \frac{\pi}{4})$ in the direction of $\mathbf{v} = \langle 1, 1\rangle$.

Solution: The gradient of f is $\nabla f(x,y) = \langle -e^{-x}\cos y, -e^{-x}\sin y\rangle$, so $\nabla f\left(0, \frac{\pi}{4}\right) = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$. The directional derivative is then

$$D_{\mathbf{u}}f\left(0, \frac{\pi}{4}\right) = \nabla f\left(0, \frac{\pi}{4}\right)\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle\cdot\frac{\langle 1,1\rangle}{\sqrt{2}} = -1.$$

Quiz 8

Name: ____________________

Math 240 - Calculus III
April 9, 2009

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (6 points) Suppose $x = v\cos u$, $y = u\sin v$, $u = t^2 - 2s$, and $v = 3t + 4s$. If $t = 0$, $s = \frac{\pi}{4}$, $\dot{t} = 2$, and $\dot{x} = 5\pi$, find $\dot{s}$ and $\dot{y}$.

   Solution: Define functions f and g: $f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v\cos u \\ u\sin v \end{pmatrix}$ and $g\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} t^2 - 2s \\ 3t + 4s \end{pmatrix}$.

   Then $\binom{x}{y} = (f \circ g)\binom{s}{t}$ and by the chain rule,

   $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[D(f \circ g)\begin{pmatrix} s \\ t \end{pmatrix}\right]\begin{pmatrix} \dot{s} \\ \dot{t} \end{pmatrix} = \left[Df\left(g\begin{pmatrix} s \\ t \end{pmatrix}\right)\right]\left[Dg\begin{pmatrix} s \\ t \end{pmatrix}\right]\begin{pmatrix} \dot{s} \\ \dot{t} \end{pmatrix}.$$

   Computing the determinant matrices:

   $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -v\sin u & \cos u \\ \sin v & u\cos v \end{bmatrix}\begin{bmatrix} -2 & 2t \\ 4 & 3 \end{bmatrix}\begin{pmatrix} \dot{s} \\ \dot{t} \end{pmatrix}.$$

   When $t = 0$ and $s = \frac{\pi}{4}$, $u = -\frac{\pi}{2}$ and $v = \pi$. Thus,

   $$\begin{pmatrix} 5\pi \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \pi & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 4 & 3 \end{bmatrix}\begin{pmatrix} \dot{s} \\ 2 \end{pmatrix} = \begin{bmatrix} -2\pi & 0 \\ 2\pi & \frac{3\pi}{2} \end{bmatrix}\begin{pmatrix} \dot{s} \\ 2 \end{pmatrix}$$

   which we solve to find $\dot{s} = -\frac{5}{2}$ and $\dot{y} = -2\pi$.

2. (4 points) What is the direction of steepest descent of $f(x, y) = x^2y - 2y$ at the point $(3, 2)$?

   Solution: The gradient of f is $\nabla f(x, y) = \langle 2xy, x^2 - 2\rangle$.

   The direction of steepest *ascent* at $(3, 2)$ is $\nabla f(3, 2) = \langle 12, 7\rangle$. Thus, the steepest *descent* is in the direction of $-\nabla f(3, 2) = \langle -12, -7\rangle$.