MEASURING FUNCTIONALS WITH HADWIGER INTEGRATION

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1. INTRODUCTION

How do we assign a notion of size to functionals? Sometimes in applications, we need to quantify how "big" a functional is. If our functional is integrable, and especially if it is compactly supported, we might say that its Lebesgue integral gives a reasonable idea of the functional's size. However, are there other equally reasonable notion of functional size? For example, an integral with respect to Euler characteristic gives a very useful concept of size in some applications. Along with Lebesgue measure and Euler characteristic, we find a family of valuations, previously defined on sets, that we can lift to real-valued functions over sets.

Euler characteristic has long been the basis of an integral calculus of integer-valued step functions. In recent years, Baryshnikov and Ghrist have successfully applied these Euler integrals to questions that arise in sensor networks. I have worked to extend the integration theory and prove new theorems. One of the most intriguing results is the following analog of the classical Hadwiger Theorem:

Theorem 1 (Baryshnikov, Ghrist, Wright)**.** *Any lower valuation v on $\mathsf{Def}(\mathbb{R}^n)$ can be written as a linear combination of lower Hadwiger integrals. For $h \in \mathsf{Def}(\mathbb{R}^n)$,*

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h) \, \lfloor d\mu_k \rfloor,$$

where the $c_k : \mathbb{R} \to \mathbb{R}$ are increasing functions with $c_k(0) = 0$.

Likewise, an upper valuation v on $\mathsf{Def}(\mathbb{R}^n)$ can be written in terms of upper Hadwiger integrals.

Open questions abound in this integration theory. There are many opportunities to prove theorems and develop algorithms that will make the integrals more useful in applications.

2. EULER INTEGRATION

The Euler characteristic[1] of a set can be interpreted as a statement about the size of a set—the number of connected components minus the number of holes. On sets, Euler characteristic is an extension of cardinality. Euler characteristic is also additive, $\chi(A \cap B) + \chi(A \cup B) = \chi(A) + \chi(B)$, the fundamental property that allows us to lift it from sets to functions over sets. Indeed, for a function $f : \mathbb{R}^n \to \mathbb{R}$ with finitely many level sets, each having a well-defined Euler characteristic, the Euler integral of f is straightforward. Write $f = \sum_i c_i \mathbf{1}_{A_i}$, and the Euler integral of f is

$$\int_{\mathbb{R}^n} f \, d\chi = \int_{\mathbb{R}^n} \sum_i c_i \mathbf{1}_{A_i} \, d\chi = \sum_i c_i \int_{\mathbb{R}^n} \mathbf{1}_{A_i} \, d\chi = \sum_i c_i \chi(A_i).$$

[1] In this paper, Euler characteristic refers to the *combinatorial* Euler characteristic: if σ is an *open* n-simplex, then $\chi(\sigma) = (-1)^n$. We extend to more general sets by additivity.

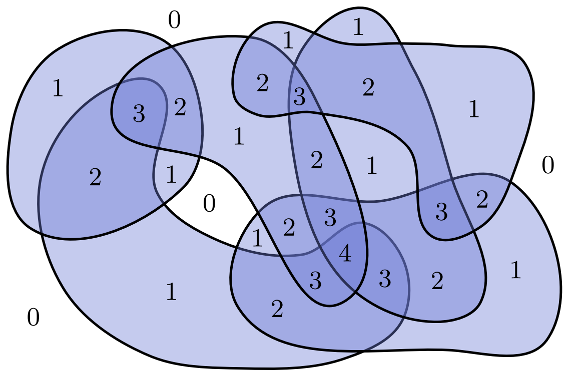

FIGURE 1. Function $h(x)$ counts the number of sets containing x.

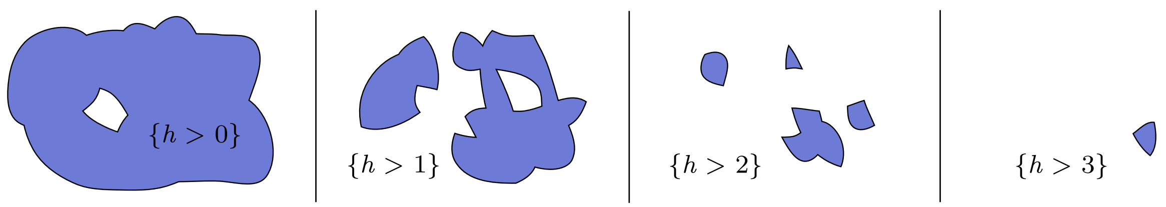

FIGURE 2. Excursion sets of h.

For an application, suppose we have a collection of possibly overlapping subsets $A_1, \ldots, A_N$ of $\mathbb{R}^2$, all of which have the same Euler characteristic μ. Also suppose we have a functional $h : \mathbb{R}^2 \to \mathbb{R}$, where $h(x)$ is the number of sets containing x. So h counts the number of subsets at each point, and we can write $h = \sum_{i=1}^N \mathbf{1}_{A_i}$. We interpret h as a function that observes the subsets locally, at each point. The Euler integral of h then gives a global count of the total number of sets:

$$\int_{\mathbb{R}^2} h \, d\chi = \int_{\mathbb{R}^2} \sum_{i=1}^N \mathbf{1}_{A_i} \, d\chi = \sum_{i=1}^N \int_{\mathbb{R}^2} \mathbf{1}_{A_i} \, d\chi = \sum_{i=1}^N \chi(A_i) = \sum_{i=1}^N \mu = N\mu.$$

Figure 1 shows several overlapping compact subsets of $\mathbb{R}^2$. Since each subset is simply connected, $\chi = 1$ for each subset. The numbers indicate the value of h in each region.

We can integrate h with respect to Euler characteristic to count the total number of subsets. Baryshnikov and Ghrist showed in [1] that the Euler integral can be computed as

$$\int_{\mathbb{R}^n} h \, d\chi = \sum_{k=0}^{\infty} \chi\{h > k\},$$

where $\{h > k\}$ is an excursion set—the set on which $h(x) > k$. Thus, the computation involves only the Euler characteristics of the excursion sets $\{h > k\}$ for $k \in \mathbb{Z}^{\geq 0}$. The excursion sets for our function h are illustrated in Figure 2. For our example, we compute the number of subsets as:

$$\int_{\mathbb{R}^2} h \, d\chi = \chi\{h > 0\} + \chi\{h > 1\} + \chi\{h > 2\} + \chi\{h > 3\} = 0 + 1 + 4 + 1 = 6.$$

A careful inspection reveals there are indeed six subsets in Figure 1. The Euler integral has taken local information (the count of sets at each point) and produced global information (the total number of sets).

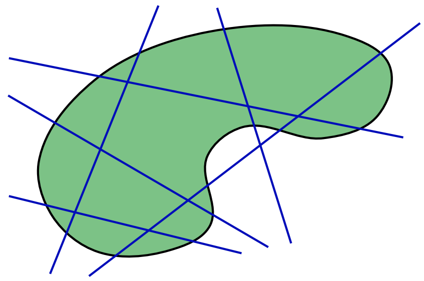

FIGURE 3. μ_1 of a subset of $\mathbb{R}^2$ is the average lengths of the intersections of all lines with the subset.

The situation is more complicated if we desire to integrate a continuous functional with respect to Euler characteristic. Baryshnikov and Ghrist have defined an Euler integral for continuous functionals, but the integral is not linear [2]. Still, it shows promise in applications and has important connections to Morse theory.

3. HADWIGER INTEGRATION

The additive property mentioned previously is the key property of Euler characteristic that allows us to use it as a measure for integration. Indeed, additivity is what allows us to call Euler characteristic a *valuation*. Generalizing Euler characteristic and Lebesgue measure, we obtain a family of valuations known as the *intrinsic volumes*.

On "tame"[2] subsets of $\mathbb{R}^n$ there exist $n+1$ valuations, invariant with respect to rigid motions, known as the intrinsic volumes.[3] We can express and compute the intrinsic volumes in various ways; for now we simply state them in terms of the Euler characteristic of all slices along affine hyperplanes:

Definition 2. The k^{th} intrinsic volume on a subset $A \subset \mathbb{R}^n$ is

$$\mu_k(A) = \int_{\mathcal{P}_{n,n-k}} \chi(A \cap P)\, d\lambda(P)$$

where $\mathcal{P}_{n,n-k}$ is the affine Grassmanian of $(n-k)$-planes in $\mathbb{R}^n$ and λ is a Haar measure on $\mathcal{P}_{n,n-k}$.

Observe that μ_0 is Euler characteristic and μ_n is Lebesgue measure. While it is more difficult to gain intuition about the other μ_k, we can regard $\mu_k(A)$ as the average k-dimensional volume of slices of A. Figure 3 illustrates an example of μ_1.

For functions on $\mathbb{R}^n$ with finitely many level sets, integration with respect to the intrinsic volumes is straightforward. We will call such an integral a *Hadwiger integral*. If $f = \sum_i c_i \mathbf{1}_{A_i}$, the k^{th} Hadwiger integral of f is

$$\mu_k(f) = \int_{\mathbb{R}^n} f\, d\mu_k = \sum_i c_i \mu_k(A_i).$$

Extending Hadwiger integration to continuous functions is more difficult, yet the difficulties provide opportunities for research. Briefly, we define *upper* and *lower* Hadwiger integrals:

[2] To properly define "tame" we would need to discuss o-minimal systems, which we will not do here. The interested reader may consult [5]. The idea is that we include nice sets and exclude pathologies such as Cantor sets.

[3] Also known as *Hadwiger measures*, *quermassintegrale*, *Lipschitz-Killing curvatures*, *Minkowski functionals*, etc.

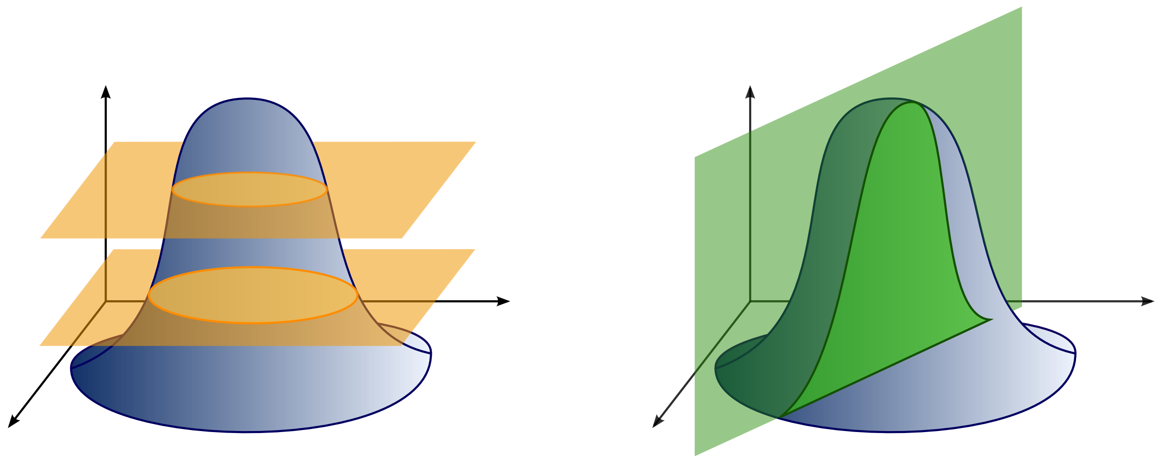

FIGURE 4. The Hadwiger integral of a function $f : \mathbb{R}^2 \to \mathbb{R}$ can be computed in terms of level sets of f (left) or slices of f by planes perpendicular to the domain (right), as in Theorem 4.

Definition 3. For tame, compactly supported $f : \mathbb{R}^n \to \mathbb{R}$, the *lower* and *upper Hadwiger integrals* of f are, respectively,

$$\int h \lfloor d\mu_k \rfloor = \lim_{m\to\infty} \frac{1}{m} \int_X \lfloor mh \rfloor \, d\mu_k \quad \text{and} \quad \int h \lceil d\mu_k \rceil = \lim_{m\to\infty} \frac{1}{m} \int_X \lceil mh \rceil \, d\mu_k.$$

These limits exist but are not equal in general. The Hadwiger integrals give many useful interpretations of the "size" of functionals. I have shown that the Hadwiger integrals can be expressed in various ways, which are useful for computations. Two expressions of the Hadwiger integrals are as follows:

Theorem 4 (Wright)**.** *The lower and upper Hadwiger integrals can be stated in terms of excursion sets or slices:*

$$\int h \lfloor d\mu_k \rfloor = \int_{s=0}^{\infty} \mu_k\{h \geq s\} - \mu_k\{h < -s\} \, ds = \int_{\mathcal{P}_{n,n-k}} \int_{X\cap P} h \lfloor d\chi \rfloor \, d\lambda(P)$$

$$\int h \lceil d\mu_k \rceil = \int_{s=0}^{\infty} \mu_k\{h > s\} - \mu_k\{h \leq -s\} \, ds = \int_{\mathcal{P}_{n,n-k}} \int_{X\cap P} h \lceil d\chi \rceil \, d\lambda(P).$$

Figure 4 illustrates Theorem 4 for a bump function. There are several other useful expressions of Hadwiger integrals, most importantly in terms of integrals over conormal cycles [3]. The *flat norm* on conormal cycles allows us to define the *lower* and *upper flat topologies* in which the lower and upper Hadwiger integrals, respectively, are continuous [6].

With an understanding of Hadwiger integrals, we can lift theorems from convex (or more generally, o-minimal) sets to functionals over sets. One of the most intriguing results concerns Hadwiger's Theorem. Classically, Hadwiger's Theorem says that any Euclidean-invariant valuation on sets, continuous on convex sets, is a linear combination of the intrinsic volumes.

A valuation on $\mathsf{Def}(\mathbb{R}^n)$ is an additive map $v : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$ that is invariant with respect to Euclidean motions of $\mathbb{R}^n$. Valuation v is a *lower* (respectively, *upper*) if v is continuous in the lower (respectively, upper) flat topology. The extension of Hadwiger's Theorem is then:

Theorem 5 (Baryshnikov, Ghrist, Wright). *Any lower valuation v on $\mathsf{Def}(\mathbb{R}^n)$ can be written as a linear combination of lower Hadwiger integrals. For $h \in \mathsf{Def}(\mathbb{R}^n)$,*

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h) \, \lfloor d\mu_k \rfloor,$$

where the $c_k : \mathbb{R} \to \mathbb{R}$ are increasing functions with $c_k(0) = 0$.

Likewise, an upper valuation v on $\mathsf{Def}(\mathbb{R}^n)$ can be written in terms of of upper Hadwiger integrals.

This analog of the classical Hadwiger Theorem provides an important classification of an entire family of valuations, each of which is an interpretation of the "size" of functionals. For more details, see my thesis [6].

4. Directions for Future Work

We have seen that Hadwiger integrals provide a mechanism for lifting valuations from sets to functionals over sets. Open questions abound in the theory of Hadwiger integration, providing opportunities for future research. Particular areas for future work include theorems, algorithms, and applications.

A primary need is for theorems to help analyze Hadwiger integrals of continuous functionals. We would like theorems to estimate such integrals, and to compare integrals of different functionals. The Steiner formula [4] provides insight into convolution of functionals involving Hadwiger integrals, which may be helpful for understanding integral transforms. The non-linearity of Hadwiger integrals of continuous functionals makes proving such theorems challenging.

Computation of Hadwiger integrals is also difficult in general, and we would like to find efficient algorithms for these computations. In applications, we often need to compute integrals of functionals given the values of the functional on only a discrete set of points. We need algorithms to perform the computation and theorems to bound the possible error in this setting.

Finally, the theory of Hadwiger integration may be applied in various real-world contexts. Baryshnikov and Ghrist have already demonstrated applications of Euler integrals in sensor networks. Applications of Hadwiger integrals may also exist in image processing and cell dynamics. Time will tell what other applications arise for this integration theory.

Hadwiger integration is an exciting new theory, involving ideas from topology, combinatorics, analysis, and other areas of mathematics, with many directions for future research.

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