Relativized Grothendieck Topoi

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Abstract

In this paper we define a notion of relativization for higher order logic. We then show that there is a higher order theory of Grothendieck topoi such that all Grothendieck topoi relativizes to all models of set theory.

1 Introduction

One of the most important properties of first order logic is that the satisfaction relationship between formulas and models is absolute. That is, given two standard set theoretic universes V_0 and V_1, a model M and formula φ of first order logic such that $M, \varphi \in V_0 \cap V_1$, we have $(M \models \varphi)^{V_0}$ if and only if $(M \models \varphi)^{V_1}$. Unfortunately though, when we move to the realm of higher order logic we often have to leave behind absoluteness of the satisfaction relation. This is because, unlike first order logic, higher order logic is able to talk about the ambient set theoretic universe. So, if we change the ambient set theoretic universe, we may change the models which satisfy a given higher order formula.

In particular, given a model M and a higher order formula φ such that $(M \models \varphi)^{V_0}$ we often won't have $(M \models \varphi)^{V_1}$ (where $V_0 \subseteq V_1$ are standard models of set theory). But, for certain φ, even if $\neg(M \models \varphi)^{V_1}$ there will be models which contain M as a subset which do satisfy φ in V_1. If there is a smallest such model, M_1, it makes sense to consider M_1 as the "relativization of M to V_1 (as a model of φ)".

In this paper we show that there is a second order theory GT whose models are exactly the definable expansions of Grothendieck Topoi (for a specific formula) and such that every model of GT has a relativization to every standard set theoretic universe (assuming the Axiom of Choice). In the process we will also show that every model of the theory of sites as well as every model of the theory of subcanonical sites has a relativization to every model of set theory.

2 Background

2.1 Set Theory

Whenever one tries to do naive category theory in the language of sets and classes one runs into a problem. This problem arises from the need not only to deal with large categories, which have a class of objects, but also to deal with categories whose objects are large categories, etc.

This paper is no different in that we will often want to deal with categories which could not reasonably exist simply under the standard axioms of set theory with classes (e.g. Zermelo-Frankel Set Theory or Godel-Berney's Set Theory). We get around this by using the method first proposed in [1]. We will work in a set theory ST whose axioms are

Definition 2.1. Let $L_{ST} = \{\in, \mathcal{S}\}$ where $\mathcal{S}$ is a constant and $\in$ is a binary relation. Let ST be the theory

- Zermelo Frankel Set Theory ([2])
- $(\exists x) x \in \mathcal{S}$
- $(\forall x, y)(y \in x \wedge x \in \mathcal{S}) \rightarrow y \in \mathcal{S}$
- $(\forall x, y)(x \in \mathcal{S} \wedge y \subseteq x) \rightarrow y \in \mathcal{S}$
- $(\forall x_1 \in s, \dots x_n \in s)(\varphi^{(s)}(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots x_n))$ for all formulas φ of L_{ST} (where $\varphi^{(\mathcal{S})}$ is the relativization of φ to $\mathcal{S}$).

We let STC be $ST + \mathcal{S} \models$"Global Axiom of Choice".

We call the elements a model of ST classes and those elements which are also elements of $\mathcal{S}$ sets.

In order to prove our results in maximum generality we will also avoid using arguments which require the Axiom of Choice whenever possible. However when a theorem does use the Axiom of Choice we will mark it with (*).

For the rest of this paper we will fix a model $Set \models ST$ and will assume that all standard models are with respect to this background model of set theory.

Definition 2.2. If $Set \models ST$ then a *standard model* is a pair of formulas formula $\varphi_{class}(x, z), \varphi_{set}(y, z)$ in L_{ST} along with a class $A \in Set$ such that $\langle \{x \in Set : \varphi_{class}(x, A)\}, \in, \{y \in Set : \varphi_{set}(y, A)\}\rangle \models ST$.

Definition 2.3. If V is a standard model and $\varphi(x, A)$ is a formula of set theory with A a set in V then $(\varphi(x, A))^V$ is the formula of set theory obtained by uniformly bounding all quantifiers by V (i.e. replacing $(\forall x)\psi(x, y)$ with $(\forall x)x \in V \rightarrow \psi(x, y)$ and replacing $(\exists x)\psi(x, y)$ with $(\exists x)x \in V \wedge \psi(x, y)$). We say $(\varphi(x, A))^V$ is the *relativization* of $\varphi(x, A)$ to V.

2.2 Models

In this paper we will be interested in models of a second order language. Given such a model we will be interested in upwards absolute extensions. That is if M is a model and $M \models R(\mathbf{a})$ then we will only be interested in models M' such that $M \subseteq M'$ and $M' \models R(\mathbf{a})$. We will not however require M' to preserve the negation of a relation unless the theory dictates (i.e. unless there is another relation R' such that the theory implies R' is equivalent to $\neg R$).

Definition 2.4. Suppose $L = \{\equiv, \not\equiv\} \cup \{R_i : i \in I\}$ where R_i is a relation with arity n_i and $\equiv, \not\equiv$ are binary relations. Models(L) is the category where

- The objects of Models(L) are sequences $\mathcal{M} = \langle M, P(M), \equiv^M, \not\equiv^M, R_i^M\rangle$ where
  - $\equiv, \not\equiv \subseteq (M \cup P(M))^2$ and $R_i \subseteq M^{n_i} \times P(M)^{m_i}$. We will use the shorthand $\mathcal{M} \models R_i(\mathbf{a})$ for $\mathbf{a} \in R_i^M$, $a \equiv b$ for $\equiv (a, b)$ and $a \not\equiv b$ for $\not\equiv (a, b)$
  - $\equiv$ is an equivalence relation on M and on $P(M)$, $\not\equiv$ is a symmetric relation on M and on $P(M)$ and $(\forall m \in M, m' \in P(M))\neg(m \equiv m') \wedge \neg(m \not\equiv m')$
  - $(\forall m_0, m_1 \in M)m_0 \not\equiv m_1 \leftrightarrow \neg(m_0 \equiv m_1)$
  - $(\forall m_0, m_1 \in P(M))m_0 \not\equiv m_1 \leftrightarrow \neg(m_0 \equiv m_1)$

  - $(\forall m, m' \in P(M)) m \equiv m' \leftrightarrow (\forall a \in M)(\exists b, b' \in M)(a \in m \to b \in m' \wedge a \equiv b) \wedge (a \in m' \to b' \in m \wedge a \equiv b')$
  - If $M \models \mathbf{a} \equiv \mathbf{b}$ then $M \models R_i(\mathbf{a}) \leftrightarrow R_i(\mathbf{b})$.

- If $\mathcal{M} = \langle M, \equiv^M, \not\equiv^M, R_i^M \rangle$ and $\mathcal{N} = \langle N, \equiv^N, \not\equiv^N, R_i^N \rangle$ then $f \in \text{Models}(L)[\mathcal{M}, \mathcal{N}]$ if $f : M \to N$ is a map such that
  - $f^* : P(M) \to P(N)$ is defined by $f^*(X) = \{f(x) : x \in X\}$
  - For all $\mathbf{a} \in M$, $\mathbf{b} \in P(M)$ if $\mathcal{M} \models R_i(\mathbf{a}, \mathbf{b})$ then $\mathcal{N} \models R_i(f``[\mathbf{a}], f^*``[\mathbf{b}])$
  - For all $a, b \in M$, if $\mathcal{M} \models a \equiv b$ then $\mathcal{N} \models f(a) \equiv f(b)$ and if $\mathcal{M} \models a \not\equiv b$ then $\mathcal{N} \models f(a) \not\equiv f(b)$
  - $(\forall m \in P(M))(\forall a \in M)[(\exists b \in M) a \equiv b \wedge b \in m] \Leftrightarrow (\exists b' \in N) f(a) \equiv b' \wedge b' \in f(m)$.

We will often associate a model with its underlying set, and omit explicit mention of $\equiv$ and $\not\equiv$ as distinguished relations.

We want to think of $\equiv$ as representing equality. As we move from one model of set theory to another the relativization of our model may have new elements. And some of these new elements may be $\equiv$ to old elements. However, because we also have $\not\equiv$ as a relation, we will never have two elements which aren't equal in a model become equal in its relativization.

It is also worth mentioning that while we want to think of elements of our model as $\equiv$-equivalence classes we can't quite do this if we don't have choice. The reason is that there may be a map between the equivalence classes of models which doesn't come from an actual map of models (because we are unable to choose a representative of our equivalence classes.)

Definition 2.5. Let $Model_=(L)$ be the full subcategory of $Models(L)$ consisting of those models $\mathcal{M} \models (\forall x, y) x \equiv y \to x = y$

Lemma 2.6. *There is a functor $F : Models(L) \to Models_=(L)$.*

Proof. For each model $\mathcal{M} = \langle M, P(M), \equiv^M, \not\equiv^M, R^M \rangle$ let $F(M) = \langle M/\equiv, P(M)/\equiv, =, \neq, R^M \rangle$ and for each $\alpha : \mathcal{M} \to \mathcal{N}$ and $a \in M$ let $F(\alpha)(a) = [a]_\equiv$. It is then clear that $F(M) \in \text{Models}(L)$ (because R^M preserves $\equiv^M$ and $P(M)/\equiv^M \cong P(M/\equiv^M)$). Further each function $F(\alpha)$ is a map between $F(M)$ and $F(N)$. □

Lemma 2.7. *F is full if the global axiom of choice holds.*

Proof. Suppose $\alpha : F(A) \to F(B) \in \text{Models}(L)$. For each $x \in F(B)$ choose an element b_x and let $\alpha^*(a) = b_x$ if and only if $\alpha([a]) = x$. Then clearly $F(\alpha^*) = \alpha$. □

Definition 2.8. If $\varphi(x, y)$ is a formula of set theory such that $Set \models \varphi(x, A) \to x \in \text{obj}(\text{Models}(L))$ then $\text{Models}(\varphi(x, A))$ is the full subcategory of $\text{Models}(L)$ containing those objects M where $Set \models \varphi(M, A)$.

2.3 Relativized Models

We will be interested in this paper with formulas and models which, as we change models of set theory, have a minimal extension.

Definition 2.9. Let $\varphi(x, A)$ be as in Definition 2.8 and let M be in $\text{obj}(\text{Models}(L))$. We then define $Ext(\varphi, M)$ to be the category whose objects are those models N such that

- $M \subseteq N$ and $R^M \subseteq R^N$ for all relations.
- $Set \models \varphi(N, A)$

and whose morphism are those maps $f \in \text{Models}(L)[N, P]$ such that $f(m) = m$ for all $m \in M$.

Definition 2.10. Suppose, $\varphi(x, A)$, M, and V_0, V_1 are such that

- V_0, V_1 are standard models of ST with $V_0 \subseteq V_1$
- $\varphi(x, A)$ is a formula for the language L with $A \in V_0$.
- $M \in \text{obj}(\text{Models}(L))$ and $(M \in \text{obj}(\text{Models}(\varphi(x, A))))^{V_0}$

If N is an initial object of $(Ext(\varphi, M))^{V_1}$ then we say N is a *relativization* of M to V_1 for $\varphi(x, A)$.

So if $M \in V_0$ the relativization of M to V_1 for $\varphi(x, A)$, if it exists, is the smallest model M' containing M such that $V_1 \models \varphi(M', A)$.

2.4 Category Theory

In this section we will review some of the categorical definitions we will need. For more information on the category theory in this section the reader is referred to such standard texts as [3]. For more information on the sheaf theoretic ideas presented in this section the reader is referred to such standard works such as [4].

All categories in this paper will be locally small and we will use the convention that if C is a category with objects A and B, $C[A,B]$ is the set of morphisms whose domain is A and whose codomain is B. We will also use $C[-,B]$ for the set of all morphisms whose codomain is B.

Definition 2.11. Let $L_{Cat} = \{Obj, Morph, Dom, Codom, Id\}$ and let $Th_{Cat}(X)$ be the formula which says

- $X \in Model(L_{Cat})$ and X is a category
- $X \models Obj(A)$ if and only A is an object of X
- $X \models Morph(f)$ if and only if f is a morphism of X
- $Dom, Codom : Morph \to Obj$ are the domain and codomain maps respectively
- $Id : Obj \to Morph$ is the map which takes an object to its identity morphism.

Definition 2.12. If C, D are categories we say that C is *isomorphic* to D ($C \cong D$) if there exists functors $i_C : C \to D$ and $i_D : D \to C$ such that $i_C \circ i_D = id_D$ and $i_D \circ i_C = id_C$. We also say that C is *equivalent* to D ($C \simeq D$) if there exists functors $e_C : C \to D$ and $e_D :\to C$ as well as natural isomorphisms $\eta_C : e_D \circ e_C \Rightarrow 1_C$ and $\eta_D : e_C \circ e_D \Rightarrow 1_D$.

Definition 2.13. Let **SET** be the category whose objects are sets in Set and whose morphisms are functions in Set.

Definition 2.14. Let C be a category with $F : C^{op} \to \mathbf{SET}$ a presheaf on C. An *element of* F is an element of $\bigcup_{A \in \mathrm{obj}(C)} F(A)$. If $f \in C[A,B], x \in F(B)$ we will use the shorthand $x|_f$ for $F(f)(x)$.

Definition 2.15. Let C be a category and $A \in \text{obj}(C)$. A *sieve* S on A is a subfunctor of $C[-, A]$. If S is a sieve on A and $f : B \to A$ then the *pullback of* S *along* f is the sieve on B given by $f^*S(D) = \{g \in C[D, B] : f \circ g \in S\}$. If $X \subseteq C[-, A]$ we define $Gen_{sieve}(X) = \{x \circ f : x \in X, \text{dom}(x) = \text{codom}(f), f \in \text{morph}(C)\}$ to be the *sieve generated by* X.

Definition 2.16. A *weak site* is a pair (C, J_C) where C is a category and J_C a function from the objects of C to collections of sieve such that for any $A \in \text{obj}(C)$:

- (Identity) $C[-, A] \in J_C(A)$
- (Base Change) If $S \in J_C(A)$ and $f : B \to A$ then $f^*S \in J_C(B)$

We call $J_C(A)$ the *covering sieves of* A

Definition 2.17. If (C, J_C) and (D, J_D) are weak sites and $F : C \to D$ is a functor then we say that F is a *map of weak sites* if $(\forall S \in J_C(A))\{F(f) \circ x : f \in S, x \in D\} \in J_D(F(A))$

Definition 2.18. A *site* is a weak site (C, J_C) satisfying

- (Local Character) Let $S \in J_C(A)$ and let T be any sieve on A. If $(\forall f \in S(B))f^*T \in J_C(B)$ then $T \in J_C(A)$.

Definition 2.19. Let (C, J_C) be a weak site and $F : C^{op} \to \textbf{SET}$ be a presheaf on C. If $A \in \text{obj}(C)$, a *compatible collection of elements on* A is a collection $\langle (a_i, i) : i \in S \rangle$ such that

- $S \in J_C(A)$.
- $(\forall i \in S(B))a_i \in F(B)$
- $(\forall i' : B' \to B)a_{i \circ i'} = F(i')(a_i)$

If there is an $a \in F(A)$ such that $a|_i = a_i$ for all $i \in S$ then we say a is *covered by* $\langle (a_i, i) : i \in S \rangle$.

Definition 2.20. If

- $X \subset C[-, A]$
- $Gen_{sieve}(X) \in J_C(A)$

Then we say $\langle (a_i, i) : i \in X\rangle$ covers a if $\langle (a_i|_f, i \circ f) : i \circ f \in Gen_{sieve}(X)\rangle$ is a cover for a.

Lemma 2.21. *If (C, J_C) is a site and $S, S' \in J_C(A)$ then $S \cap S' \in J_C(A)$*

Proof. First if $f \in S \cap S'$ then so is $f \circ x$ for any x. Hence $S \cap S'$ is a sieve if S and S' are. Also, for all $f \in S$ and all x we have $f \circ x \in S$. So if $f \in S$, $f^*S' \subseteq f^*S$ and hence $f^*(S \cap S') = f^*S' \in J_C(A)$. So by (Local Character) $S \cap S' \in J_C(A)$. □

Definition 2.22. Suppose (C, J_C) is a weak site. A presheaf $F : C^{op} \to$ **SET** is *separated for* (C, J_C) if for every compatible collection of elements $\langle (a_i, i) : i \in S\rangle$ on A there is a most one $a \in F(A)$ covered by $\langle (a_i, i) : i \in S\rangle$. We let **Sep**$(C, J_C)$ be the category whose objects are separated presheaves for (C, J_C) and whose morphisms are natural transformations.

Definition 2.23. Suppose (C, J_C) is a weak site. A presheaf $F : C^{op} \to$ **SET** is a *sheaf for* (C, J_C) if for every compatible collection of elements $\langle (a_i, i) : i \in S\rangle$ on A there is exactly one $a \in F(A)$ covered by $\langle (a_i, i) : i \in S\rangle$. We let **Sheaf**$(C, J_C)$ be the category whose objects are sheaves for (C, J_C) and whose morphisms are natural transformations.

Lemma 2.24. *Suppose (C, J_C) is a site, F is a separated presheaf for (C, J_C) and $S, S' \in J_C(A)$ with $S \subseteq S'$. If $\langle (a_i, i) : i \in S'\rangle, \langle (b_i, i) : i \in S'\rangle$ are compatible collections of elements of F such that $(\forall j \in S)a_j = b_j$ then $(\forall j \in S')a_j = b_j$.*

Proof. For all $j \in S'$, $\langle a_{j \circ k} : k \in j^*S\rangle$ is a cover of a_j and b_j. Hence, because F is separated, $a_j = b_j$. □

Definition 2.25. Suppose (C, J_C) is a weak site and F is a presheaf for (C, J_C). We say $X \subseteq \bigcup_{A \in \mathrm{obj}(C)} F(A)$ *covers* F if for all $A \in \mathrm{obj}(C)$ and for all $a \in F(A)$ there exists $\langle (a_i, i) : i \in S\rangle$ such that

- $\langle (a_i, i) : i \in S\rangle$ covers a
- $(\forall i \in S(B))(\exists \alpha :\in C[B, B'])(\exists b \in X \cap F(B'))b|_\alpha = a_i$

We say that a subpresheaf $G \rightarrowtail F$ *covers* F if $\bigcup_{A \in \mathrm{obj}(C)} G(A)$ covers F.

A set covers F if every other element of F can be covered by restrictions of elements of X. So in particular F can be recovered from X (and the site).

Lemma 2.26. *Suppose F is a separated presheaf for (C, J_C) and X covers F. Then there is a smallest subpresheaf $F_X \subseteq F$ such that $X \subseteq \bigcup_{A \in obj(C)} F(A)$ and F_X covers F.*

Proof. Let $F_X(A) = \{a : (\exists f \in C[A, B])(\exists b \in F(B) \cap X)F(f)(b) = a\}$. F_X is clearly the smallest subpresheaf of F containing X (and so F_X covers F). □

Lemma 2.27. *If (C, J_C) is a site and $F \subseteq G \subseteq H$ are separated presheaves for (C, J_C) such that F covers G and G covers H then F covers H.*

Proof. This follows from (Local Character). □

Notice this does not hold if (C, J_C) is only a weak site.

Lemma 2.28. *Suppose (C, J_C) is a site, F is a separated presheaf for (C, J_C) and F_0, F_1 covers for F. Then $G(A) = F_0(A) \cap F_1(A)$ is also a covering presheaf for F.*

Proof. By (Local Character) and Lemma 2.27 it suffices to prove that every element of $F_0(A)$ is covered by a collection of elements in $F_0 \cap F_1$. Every element a of F_0 is covered by a collection of elements, $\langle (a_i, i) : i \in S \rangle$, from F_1 (as $F_0 \subseteq F$). But any cover of a must consist of restrictions of a. Hence any cover must consist of elements of F_0. In particular we have the cover $\langle (a_i, i) : i \in S \rangle$ consists of elements of $F_0 \cap F_1$. □

Definition 2.29. Let $y_C : C \to \mathbf{SET}^{C^{op}}$ be the *Yoneda Embedding*. i.e. $y_C(A) = C[-, A]$ for all $A \in \text{obj}(C)$ and $y_C(f) = C[-, f]$ for all $f \in \text{morph}(C)$.

Definition 2.30. We say a site (C, J_C) is *subcanonical* if $y_C(A)$ is a sheaf for all $A \in \text{obj}(C)$. We say (C, J_C) is *almost subcanonical* if $y_C(A)$ is a separated presheaf for all $A \in \text{obj}(C)$.

Theorem 2.31. *Suppose V_0, V_1 are standard models of set theory with $V_0 \subseteq V_1$. Further suppose that $(C, J_C) \in V_0$ and $V_0 \models$ "(C, J_C) is an almost subcanonical weak site". Then $V_1 \models$ "(C, J_C) is an almost subcanonical weak site".*

Proof. Suppose (in V_1) $A \in \text{obj}(C)$, $f, g \in y_C(A)(B)$ and $\langle (x_i, i) : i \in S \rangle$ with $S \in J_C(B)$ such that both f and g are covered by $\langle (x_i, i) : i \in S \rangle$. Then the same holds in V_0 and hence $f = g$. So (in V_1) (C, J_C) is almost subcanonical. □

Almost subcanonical sites are the absolute analog of subcanonical sites.

3 Sites

Before we begin discussing relativizations of Grothendieck Topoi we will discuss relativizations of sites. In Section 3.1 we will show that every model of the theory of sites has a relativization every standard model of set theory. Then, in Section 3.2, we will show that every model of the theory of subcanonical sites has a relativization for every model of set theory.

It is worth noting that these are two different theories and even when a single site satisfies both of them (in a given model of set theory) that does not mean that the relativization of that site for these two theories will be the same.

3.1 Relativized Sites

Definition 3.1. Let (C, J_C) be a weak site. Define J_C^α on $A \in \mathrm{obj}(C)$ as follows

- $J_C^0(A) = J_C(A)$
- $J_C^{\alpha+1}(A) = \{T \text{ a sieve on } A : (\exists S \in J_C(A))(\forall f \in S(B)) f^*T \in J_C^\alpha(B)\}$.
- $J_C^{\omega\cdot\gamma}(A) = \bigcup_{\beta<\omega\cdot\gamma} J_C^\beta(A)$

We define $J_C^{\mathrm{ORD}} = \bigcup_{\alpha\in\mathrm{ORD}} J_C^\alpha$ and if $T \in J_C^{\mathrm{ORD}}(A)$ we say the *degree* of T is the least ordinal α such that $T \in J_C^\alpha(A)$.

Theorem 3.2. *(C, J_C^{ORD}) is a Site.*

Proof. (Identity):
This is immediate because $J_C \subseteq J_C^{\mathrm{ORD}}$.

(Local Character):
Let $S \in J_C^{\mathrm{ORD}}(A)$ and let T be any sieve on A such that $f^*T \in J_C^{\mathrm{ORD}}(B)$ for all $f \in S$. We want to show $T \in J_C^{\mathrm{ORD}}(A)$. If degree$(S) = 0$ then $T \in J_C^\alpha(A)$ where $\alpha = \sup\{degree(f^*T) : f \in S\}$.

Assume that the conclusion in (Local Character) holds if degree$(S) \leq \alpha$ and let degree$(S) = \alpha + 1$. Then there is an $R \in J_C(A)$ such that $(\forall f \in R(B)) f^*S \in J_C^\alpha(B)$. Now if $f \in R$ then f^*T is a sieve on dom(f) and $f^*S \in J_C^\alpha(\mathrm{dom}(f))$. But $(\forall g \in f^*S) f \circ g \in S$ and $g^*f^*T = (f \circ g)^*T$.

So $g^*f^*T \in J_C^{\mathrm{ORD}}(\mathrm{dom}(g))$. Hence by the inductive assumption, $f^*T \in J_C^{\mathrm{ORD}}(\mathrm{dom}(f))$ for all $f \in R$, and by the definition of J_C^{ORD}, $T \in J_C^{\mathrm{ORD}}(A)$. So (C, J_C^{ORD}) satisfies (Local Character).

(Change of Base):
Let $g : D \to A$ and $T \in J_C^{\mathrm{ORD}}(A)$. If degree$(T) = 0$ then $g^*T \in J_C^{\mathrm{ORD}}(D)$ because $J_C \subseteq J_C^{\mathrm{ORD}}$.

Assume the conclusion in (Change of Base) holds if degree$(T) \leq \alpha$ and let degree$(T) = \alpha + 1$. Then there is a cover $S \in J_C(A)$ such that $(\forall f \in S(B)) f^*T \in J_C^{\alpha}(B)$. Now $g^*S \in J_C(B)$. So $(\forall h \in g^*S(D)) g \circ h \in S(D)$ and $(g \circ h)^*T = h^*(g^*T) \in J_C^{\alpha}(D)$. Hence $g^*T \in J_{\alpha+1}(B)$. So (C, J_C^{ORD}) satisfies (Change of Base). □

Corollary 3.3. *If (C, J_C) is a weak site then so is (C, J_C^{α}) for all $\alpha \leq ORD$.*

Definition 3.4. Let $L_{Weak\ Site} = L_{Cat} \cup \{Covers(x, y)\}$ where $Covers(x, y)$ is a binary relation, and $Th_{Weak\ Site}(X)$ be the formula which says

- If $C = X|_{L_{Cat}}$ then $Th_{Cat}(C)$
- $Covers(S, A) \to S \in P(X), A \in X$
- If we define $S \in JC(A)$ to be a short hand for $Covers(S, A)$ then (C, J_C) is a weak site.

We will consider all weak sites (C, J_C) as models of the language $L_{Weak\ Site}$ using the interpretation of $Cover(S, A) \leftrightarrow S \in J_C(A)$.

Definition 3.5. Let $L_{Site} = L_{Weak\ Site}$ and let $Th_{Site}(X)$ be the formula which says

- $Th_{Weak\ Site}(X)$
- If $C = X|_{L_{Cat}}$ and $S \in J(A)$ is a short hand for $Covers(S, A)$ then (C, J_C) is a site.

Theorem 3.6. *If (C, J_C) is a weak site then (C, J_C^{ORD}) is an initial object in $Ext(Th_{Site}, (C, J_C))$.*

Proof. By Theorem 3.2 $(C, J_C^{\mathrm{ORD}}) \models Th_{Site}$. However, if $(D, J_D) \in \mathrm{obj}(Ext(Th_{Site}, (C, J_C)))$ then C is a subcategory of D and $J_C \subseteq J_C^{\mathrm{ORD}}$. So the (unique) map from (C, J_C^{ORD}) into (D, J_D) mapping C to itself is a map of weak sites. Hence (C, J_C^{ORD}) is an initial object in $Ext(Th_{Site}, (C, J_C))$. □

In particular, given any site (C, J_C), there is a relativization of (C, J_C) for Th_{Site} to any model of set theory.

Not surprisingly we find that the property of being a separated presheaf or of being sheaf is independent of whether or not we consider the a weak site (C, J_C) or its relativization (C, J_C^{ORD}).

Theorem 3.7. *Suppose F is a presheaf on C. Then*

(a) F is separated for (C, J_C) if and only if F is separated for (C, J_C^{ORD}).

(b) F is a sheaf for (C, J_C) if and only if F is a sheaf for (C, J_C^{ORD}).

Proof. Part (a):
First it is clear that if F is separated for (C, J_C^{ORD}) then F is separated for (C, J_C) as $J_C \subseteq J_C^{\mathrm{ORD}}$.

For the other direction lets assume, to get a contradiction, that F is separated for (C, J_C) but not separated for (C, J_C^{ORD}). Then there is a $\langle (a_i, i) : i \in S\rangle$ with $S \in J_C^{\mathrm{ORD}}(A)$ and there are $\overline{a}_0, \overline{a}_1 \in F(A)$ such that $\overline{a}_0$ and $\overline{a}_1$ are covered by $\langle (a_i, i) : i \in S\rangle$. Assume that S has minimal degree of $\alpha+1$ such that the above is true (it can't have degree 0 as F is separated for $(C, J_C) = (C, J_C^0)$). Then there is an $T \in J_C(A)$ such that for all $f \in T(B)$, $f^*S \in J_C^{\alpha}(B)$. So in particular $\overline{a}_0|_f = \overline{a}_1|_f$ for all $f \in T$ as $\overline{a}_0|_f, \overline{a}_1|_f$ are both covered by $\langle (a_{f\circ i}, i) : i \in f^*S\rangle$ and f^*S has degree less than $\alpha + 1$. But we also have $\overline{a}_0, \overline{a}_1$ are covered by $\langle (a_0|_f, f) : f \in T\rangle$ (because it is a compatible collection) and hence $\overline{a}_0 = \overline{a}_1$ as T has degree 0.

Part (b):
First it is clear that if F is a sheaf for (C, J_C^{ORD}) then F is also a sheaf for (C, J_C), as $J_C \subseteq J_C^{\mathrm{ORD}}$.

For the other direction assume, to get a contradiction, that F is a sheaf for (C, J_C) but not a sheaf for (C, J_C^{ORD}). By Part (a) we know that F is separated for (C, J_C^{ORD}). So there must be a collection $\langle (a_i, i) : i \in S\rangle$ with $S \in J_C^{\mathrm{ORD}}(A)$ which is compatible but which doesn't cover any element of F. Let S have minimal degree of $\alpha + 1$ such that the above is true (it can't have degree 0 as F is a sheaf for $(C, J_C) = (C, J_C^0)$). Then there is a $T \in J_C(A)$ such that for all $f \in T(B)$, $f^*S \in J_C^{\alpha}(B)$. So in particular $\langle (a_{f\circ g}|g, g) : g \in f^*S\rangle$, with $S \in J_C^{\mathrm{ORD}}(A)$, is a compatible collection of elements for all $f \in T$ and hence must cover an element $a_f \in F(\mathrm{dom}(f))$ by the inductive hypothesis. But the collection $\langle (a_f, f) : f \in T\rangle$ is also compatible and hence must cover an element $a \in A$ also by the inductive hypothesis. And, as this a must also be covered by $\langle (a_i, i) : i \in S\rangle$ by construction, we have our contradiction. □

Corollary 3.8. *If (C, J_C) is an almost subcanonical weak site then so is (C, J_C^{ORD}).*

Corollary 3.9. *If (C, J_C) is a weak site then $\boldsymbol{Sep}(C, J_C) = \boldsymbol{Sep}(C, J_C^{\boldsymbol{ORD}})$ and $\boldsymbol{Sheaf}(C, J_C) = \boldsymbol{Sheaf}(C, J_C^{\boldsymbol{ORD}})$.*

Theorem 3.10. *There is a functor $\boldsymbol{a} : \boldsymbol{Sep}(C, J_C) \to \boldsymbol{Sheaf}(C, J_C)$ such that if $\boldsymbol{i} : \boldsymbol{Sheaf}(C, J_C) \to \boldsymbol{Sep}(C, J_C)$ is the inclusion functor then $\boldsymbol{i}$ is left adjoint to $\boldsymbol{a}$ and the unit $\iota : 1_{\boldsymbol{Sep}(C,J_C)} \Rightarrow \boldsymbol{a} \circ \boldsymbol{i}$ is such that for all $F \in obj(\boldsymbol{Sep}(C, J_C))$ and all $A \in obj(C)$, $(\iota_F)_A$ is the identity on its domain. We call $\boldsymbol{a}$ the sheafification functor.*

Proof. Let $F^*(A) = \{\langle (b_i, i) : i \in S\rangle : S \in J_C^{\mathrm{ORD}}(A), \langle (b_i, i) : i \in S\rangle$ a compatible collection of elements$\}$ and let $\langle (b_i, i) : i \in S\rangle \sim \langle (b'_j, j) : j \in S'\rangle$ if $(\forall i \in S \cap S')b_i = b'_i$. If $S, S' \in J_C^{\mathrm{ORD}}(A)$ then $S \cap S' \in J_C^{\mathrm{ORD}}(A)$ and so for all $i \in S$, b_i is covered by $\langle (b_{i\circ k}, k) : k \in i^*(S \cap S')\rangle$. Hence if $\langle (b_i, i) : i \in S\rangle \sim \langle (b'_j, j) : j \in S'\rangle$ and $\langle (b'_i, i) : i \in S\rangle \sim \langle (b'_j, j) : j \in S'\rangle$ then $b_i = b'_i$ for all $i \in S$ and in particular $\sim$ is an equivalence relation.

For each $\langle (b_i, i) : i \in S\rangle$ let $\overline{\langle (b_i, i) : i \in S\rangle} = \bigcup\{\langle (b'_i, i) : i \in S'\rangle : \langle (b'_i, i) : i \in S'\rangle \sim \langle b_i : i \in S\rangle\}$. Then by the previous paragraph, $\langle (b_i, i) : i \in S\rangle \sim \overline{\langle (b_i, i) : i \in S\rangle}$. So we can let $\mathbf{a}(F)(A) = F(A) \cup \{\overline{\langle b_i : i \in S\rangle} : \langle b_i : i \in S\rangle$ does not cover any element of $F(A)\}$. If $F, G \in \mathbf{Sep}(C, J_C)$ and $f : \mathbf{Sep}(C, J_C)[F, G]$ then $\mathbf{a}(f)$ is defined so that

- If $b \in F(A)$ then $\mathbf{a}(f)(b) = f(b)$
- $\mathbf{a}(f)(\langle (b_i, i) : i \in S\rangle) = \langle (f(b_i), i) : i \in S\rangle$

□

Obviously it is possible to define a sheafification functor on any presheaves and not just those which are separated. However the point of Theorem 3.10 is that we are defining sheafification in such a way that there is always inclusion map from the presheaf into the sheaf which is the identity on its domain.

3.2 Subcanonical Sites

Lemma 3.11. *Suppose* (C, J_C) *is an almost subcanonical weak site. Then* $(\forall S \in J_C^{ORD}(A))S$, *S covers* $y_C(A)$.

Proof. Suppose $x \in y_C(A)(B)$ and $S \in J_C^{\mathrm{ORD}}(A)$. Then $x \in C[B, A]$ and x^*S is a cover of B. Hence $\langle (x \circ i, i) : i \in x^*S\rangle \subseteq S$ is a compatible collection of elements which cover x. So S is a covering subsheaf for $y_C(A)$ (as x was arbitrary). □

Theorem 3.12. *Let* $Th_{SubCan}(X) = Th_{Site}(X) \cup \{X \textit{ is a subcanonical site}\}$. *Then for any almost subcanonical site* (C, J_C)[1] , $Ext(Th_{SubCan}, (C, J_C))$ *has an initial object.*

Proof. First we need to show that $(\exists X)X \in \mathrm{obj}(Ext(Th_{SubCan}, (C, J_C)))$. Define $X' = (\overline{C}, J_{\overline{C}})$ such that

- $\mathrm{obj}(C) = \mathrm{obj}(\overline{C})$
- $\overline{C}[A, B] = \{\langle (b_i, i) : i \in S\rangle$ such that $S \in J_C(A)$ and $(\forall i \in S)b_i : \mathrm{dom}(i) \to B, b_{i\circ j} = b_i \circ j\}/\equiv$
- $\langle (b_i, i) : i \in S\rangle \equiv \langle (b'_i, i) : i \in S'\rangle$ if and only if $b_i = b'_i$ for all $i \in S \cap S'$.

That $\equiv$ is an equivalence relation follows immediately from Lemma 2.21 and Lemma 2.24.

If $\langle (b_i, i) : i \in S\rangle \in \overline{C}[A, B]$ and $\langle (d_i, i) : i \in S'\rangle \in \overline{C}[B, D]$ we define $\langle (d_j, j) : j \in S'\rangle \circ \langle (b_i, i) : i \in S\rangle = \langle (d_{b_i}, i) : i \in S"\rangle$ where $S" = \{i : b_i \in S'\}$. In order to show that this definition makes sense, we first need to show that $\langle (d_{b_i}, i) : i \in S"\rangle \in \overline{C}[A, D]$. Or, more specifically, that $S" \in J_C(A)$. But we know that for all $i \in S$, $i^*S" = \{\alpha : i \circ \alpha \in S"\} = \{\alpha : b_{i\circ\alpha} \in S'\} = \{\alpha : b_i \circ \alpha \in S'\} = b_i^*S' \in J_C(\mathrm{dom}(b_i))$. Hence $S" \in J_C(A)$ by (Local Character).

To show that composition is well defined we need to show that composition is closed under $\equiv$. It suffices to show that if $S_0 \subseteq S$, $S'_0 \subseteq S'$ then $\langle (d_{b_i}, i) : i \in S_0"\rangle = \langle (d_j, j) : j \in S'_0\rangle \circ \langle (b_i, i) : i \in S_0\rangle \equiv \langle (d_j, j) : j \in$

[1]If we don't require our site to be almost subcanonical then there will non-equal $\equiv$-equivalence classes which are covered by the same cover. So, in any subcanonical extension those $\equiv$-equivalence classes would have to be identified. But, because both $\equiv$ and $\not\equiv$ are relations in our models we can't identify distinct $\equiv$-equivalence classes in extensions.

$S'\rangle \circ \langle (b_i, i) : i \in S\rangle = \langle (d_{b_i}, i) : i \in S"\rangle$. But it is clear that $S_0" \subseteq S"$ by construction and hence $S_0" \cap S" = S_0"$. So $\langle (d_{b_i}, i) : i \in S_0"\rangle \equiv \langle (d_{b_i}, i) : i \in S"\rangle$.

We have shown that $\overline{C}$ is a well defined category. In general though we don't have $C \subseteq \overline{C}$. But what we do have is a injective function $f : C \to \overline{C}$ where f is the identity on objects and $f(\alpha) = \langle (\alpha \circ i, i) : i \in C[-, \mathrm{dom}(\alpha)]\rangle$. Further $f(\alpha \circ \beta) = \langle (\alpha \circ \beta \circ i, i) : i \in C[-, \mathrm{dom}(\alpha \circ \beta)]\rangle = f(\alpha) \circ f(\beta)$. Hence f is a functor. Further if $f(\alpha) \equiv f(\beta)$ then there is a covering sieve S on $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ such that $(\forall i \in S)\alpha \circ i = \beta \circ i$ and hence $f(\alpha)$ covers α and β. But then as (C, J_C) is almost subcanonical $\alpha = \beta$. So f is injective (up to $\equiv$).

If we let $X = (X' - image(f)) \cup C$ then we have an isomorphism between X and X' which is the identity on $(X' - image(f))$ and f on C. We define composition on X so as to make this an isomorphism of categories.

All that remains is to define the collection of covering sieves. To simplify notation we will work with X' and define the covering sieves on X to be those which are the images of covering sieves on X' under the isomorphism.

Definition 3.13. For a map $\mathbf{f} = \langle (f_i, i) : i \in T\rangle$ we say $f \in \mathbf{f}$ if there is an $i \in T$ such that $f = f_i$. For a sieve S we say $S \in J_{\overline{C}}(A)$ if and only if $\widetilde{S} = \{a : (\exists \mathbf{a} \in S) a \in \mathbf{a}\} \in J_C(A)$.

Given a sieve S on C we will often want to consider the sieve it generates in $\overline{C}$. As such we will let $S_{\overline{C}} = \{f \circ x : f \in S, x \in \overline{C}\}$.

Claim 3.14. *If $\langle (g_j, j) : j \in S\rangle \in \overline{C}[A, B]$ and $i \in S(D)$ then $\langle (g_j, j) : j \in S\rangle \circ i \equiv g_i$.*

Proof. Let $\mathbf{i} = f(i) = \langle (i \circ \alpha, i), \alpha \in C[-, D]\rangle$. We know that $\langle (g_j, j) : j \in S\rangle \circ i \equiv \langle (g_i, i) : i \in S\rangle \circ \mathbf{i} = \langle (g_{i\circ\alpha}, \alpha) : i \circ \alpha \in S\rangle = \langle (g_i \circ \alpha, \alpha) : \alpha \in i^* S\rangle \equiv g_i$ □

Corollary 3.15. *If S is a sieve on $(\overline{C}, J_{\overline{C}})$ then $\widetilde{S} = S \cap morph(C)$.*

Claim 3.16. *If $\langle (g_j, j) : j \in S\rangle \in \overline{C}[A, B]$ and $i \in C[B, D]$ then $i \circ \langle (g_j, j) : j \in S\rangle \equiv \langle (i \circ g_j, j) : j \in S\rangle$.*

Proof. Let $\mathbf{i} = f(i) = \langle (i \circ \alpha, i), \alpha \in C[-, D]\rangle$. We know that $i \circ \langle (g_j, j) : j \in S\rangle \equiv \mathbf{i} \circ \langle (g_j, j) : j \in S\rangle = \langle (i \circ g_j, j) : g_j \in C[-, D]\rangle = \langle (i \circ g_j, j) : j \in S\rangle$ □

Claim 3.17. *If $(\overline{C}, J_{\overline{C}})$ is a subcanonical site $(\overline{C}, J_{\overline{C}})$ is an initial element of $Ext(Th_{SubCan}, (C, J_C))$.*

Proof. If $S \in J_{\overline{C}}(A)$ and $\langle (f_i, i) : i \in S' \rangle \in S(B)$ then for each $i \in S'$ $\langle (f_i, i) : i \in S' \rangle \circ i = f_i$ and hence $\widetilde{S}_{\overline{C}} \subseteq S$. So in particular site on $\overline{C}$ containing J_C must contain $J_{\overline{C}}$ because if $S \in J_{\overline{C}}(A)$ and $S \subseteq S'$ then $S' \in J_{\overline{C}}(A)$. □

Claim 3.18. *If $S \in J_{\overline{C}}(A)$ and $S \subseteq S'$ then $S' \in J_{\overline{C}}A$.*

Proof. We have $\widetilde{S} \subseteq \widetilde{S}'$ and $\widetilde{S} \in J_C(A)$ and so $\widetilde{S}' \in J_C A$ (because (C, J_C) is a site). □

Claim 3.19. *$(\overline{C}, J_{\overline{C}})$ is a site.*

Proof. It is clear that $J_{\overline{C}}$ satisfies (Identity).

Suppose $S \in J_{\overline{C}}(B)$ and T is a sieve on B such that $\mathbf{f}^* T \in J_{\overline{C}}(\mathrm{dom}(\mathbf{f}))$ for all $\mathbf{f} \in S$ and hence $\widetilde{\mathbf{f}^* T} \in J_C(\mathrm{dom}(\mathbf{f}))$. If $f \in \widetilde{S}$ then $f^* \widetilde{T} = \{g : f \circ g \in \widetilde{T}\}$. But we also have $f^* T = \{\langle g_i : i \in W \rangle : f \circ \langle g_i : i \in W \rangle \in T\} = \{\langle g_i : i \in W \rangle : \langle f \circ g_i : i \in W \rangle \in T\}$. So if $g \in \widetilde{f^* T}$ then $f \circ g \in \widetilde{T}$ and $g \in f^* \widetilde{T}$. Hence $\widetilde{f^* T} \subseteq f^* \widetilde{T}$ and so $f^* \widetilde{T} \in J_C(A)$. But then, because (C, J_C) satisfies (Local Character) we have $\widetilde{T} \in J_C(B)$ and hence $T \in J_{\overline{C}}(B)$. So $(\overline{C}, J_{\overline{C}})$ satisfies (Local Character).

Suppose $S \in J_{\overline{C}}(B)$ and let $\mathbf{f} = \langle f_i : i \in T \rangle \in \overline{C}[A, B]$. Then for all $i \in T$,

$$\begin{aligned} i^*(\widetilde{\mathbf{f}^* S}) &= \{g : i \circ g \in \widetilde{\mathbf{f}^* S}\} \\ &= \{g \in \mathrm{morph}(C) : i \circ g \in \mathbf{f}^* S\} \\ &= \{g \in \mathrm{morph}(C) : \mathbf{f} \circ i \circ g \in S\} \\ &= \{g \in \mathrm{morph}(C) : f_i \circ g \in S\} \\ &= f_i^* \widetilde{S} \end{aligned}$$

But $f_i^* \widetilde{S} \in J_C(\mathrm{dom}(f_i))$ because $S \in J_{\overline{C}}(B)$ and hence $\widetilde{S} \in J_C(B)$. So $i^*(\widetilde{\mathbf{f}^* S}) \in J_C(\mathrm{dom}(i))$ for all $i \in T$. Hence, by (Local Character) we have $\widetilde{\mathbf{f}^* S} \in J_C(\mathrm{dom}(\mathbf{f}))$ and therefore $\mathbf{f}^* S \in J_{\overline{C}}(B)$. So $(\overline{C}, J_{\overline{C}})$ satisfies (Base Change) and hence is a site. □

Claim 3.20. *$(\overline{C}, J_{\overline{C}})$ is a subcanonical site.*

Proof. First notice that if $\boldsymbol{\beta} = \langle (\beta_i, i) : i \in S' \rangle, \boldsymbol{\gamma} = \langle (\gamma_i, i) : i \in S" \rangle \in C[B, A]$ are such that $\boldsymbol{\beta}|_j = \boldsymbol{\gamma}_j$ for all $j \in S$ (with $S \in J_{\overline{C}}$) then the same holds for all $j \in \widetilde{S}$. So in particular for all $j \in \widetilde{S} \cap S' \cap S$" $\beta_j = \gamma_j$. So, because C is almost subcanonical, we have that $\beta_j = \gamma_j$ for all $j \in S' \cap S$" as β_j and γ_j are both covered by $\langle(\beta_j \circ i, i) : i \in j^*\widetilde{S}\rangle$. Hence $\boldsymbol{\beta} \equiv \boldsymbol{\gamma}$. So $y_{\overline{C}}(A)$ is a separated presheaf (and $(\overline{C}, J_{\overline{C}})$ is almost subcanonical).

Suppose $S \in J_{\overline{C}}(B)$ and $\langle(\alpha_i, i) : i \in S\rangle$ is a compatible collection of elements in $y_{\overline{C}}(A)$. Then $\langle(\alpha_i, i), i \in \widetilde{S}\rangle$ is a compatible collection and $\overline{S} \in J_C(B)$

If $i \in \widetilde{S}(D_i)$ then $\alpha_i \in y_{\overline{C}}(A)(D_i)$ and hence $\alpha_i : D_i \to A$. Now if $j \in C[D_k, D_i]$ and $k = i \circ j$ then $\alpha_k \equiv \alpha_i|_j = \alpha_i \circ j$. But if $\alpha_i = \langle f_{i,n} = n \in T_i\rangle$ then $\alpha_i \circ j = \langle f_{i,j\circ n} : n \in j^*T_i\rangle$. So $(\forall n \in j^*T_i \cap T_j) f_{k,n} = f_{i,j\circ n}$. Further, if $\overline{T} = \langle k \circ n : n \in T_k\rangle$ then $\overline{T}$ is a sieve and for all $i \in S$ $T_i \subseteq i^*\overline{T}$, and so $i^*\overline{T} \in J_C(D_i)$ and hence, by (Local Character), $\overline{T} \in J_C(B)$.

So, if $\alpha = \langle(f_{k,n}, k \circ n\rangle : k \circ n \in \overline{T}\rangle$ then $\alpha \in y_{\overline{C}}(A)(B)$ and $\alpha|_i = \alpha_i$. Hence $y_{\overline{C}}(A)$ is a sheaf and $(\overline{C}, J_{\overline{C}})$ is subcanonical. □

□

Theorem 3.21. *For any almost subcanonical weak site* (C, J_C)*,* $Ext(Th_{SubCan}, (C, J_C))$ *has an initial object.*

Proof. This is because if (C, J_C) is an almost subcanonical weak site then (C, J_C^{ORD}) is an almost subcanonical site such that for any site (D, J_D) containing (C, J_C) there is a unique map (C, J_C^{ORD}) into (D, J_D). □

Notice that this only works if our site is almost subcanonical to start with. Otherwise in the sheafification process we have to make different maps become the same and hence we loose preservation of "not equals". And, if we don't require preservation of not equals, there are other ways we could turn (C, J_C) into a subcanonoical site (e.g. we could collapse all morphisms and turn the resulting partial order into a subcanonical site. This might be minimal (depending on J_C) because even though we have added new elements to the $\equiv$ relation we might be able to get away with adding fewer new covers than we otherwise could).

Definition 3.22. Suppose (C, J_C) and (D, J_D) are weak sites. We say that $F : C \to D$ is an *equivalence of sites* for of (C, J_C) and (D, J_D) if

- F is an equivalence of categories.

- For all $A \in \text{obj}(C)$ and for all sieves S on A, $S \in J_C(A) \leftrightarrow \overline{F\text{"}[S]} \in J_D(F(A))$ where $\overline{F\text{"}[S]} = \{F(f) \circ g : f \in S, g \in \text{morph}(D)\}$ is the sieve generated by $F\text{"}[S]$.

Lemma 3.23. *If $E : C \to D$ is an equivalence of sites for (C, J_C) and (D, J_D) and $E' : D \to C$ is a functor such that $E \circ E' \simeq 1_D$ and $E' \circ E \simeq 1_C$ then E' is an equivalence of sites for (D, J_D) and (C, J_C).*

Proof. Immediate. □

Theorem 3.24. *Suppose (C, J_C) and (D, J_D) are weak almost subcanonical sites with $(\overline{C}, J_{\overline{C}})$ and $(\overline{D}, J_{\overline{D}})$ the corresponding initial elements of $Ext(Th_{SubCan}, (C, J_C))$ and $Ext(Th_{SubCan}, (D, J_D))$ respectively. If $E : C \to D$ is an equivalence of sites then E extends to an equivalence of sites $\overline{E} : \overline{C} \to \overline{D}$. Where $\overline{E}(\langle(\alpha_i, i) : i \in S\rangle) = \langle(E(\alpha_i) \circ g, E(i) \circ g) : i \in S, g \in D\rangle$.*

Proof. This follows immediately from the definition of $(\overline{C}, J_{\overline{C}})$ and $(\overline{D}, J_{\overline{D}})$. □

Definition 3.25. Suppose (C, J_C) is a weak almost subcanonical site. Let $\overline{(C, J_C)}$ be the full subcategory of **Sheaf**(C, J_C) whose objects are $\{y_C(A) : A \in \text{obj}(C)\}$.

Theorem 3.26. *Suppose (C, J_C) is a weak almost subcanonical site with $(\overline{C}, J_{\overline{C}})$ the corresponding initial element of $Ext(Th_{SubCan}, (C, J_C))$. Then $\overline{(C, J_C)}$ is isomorphic to $(\overline{C}, J_{\overline{C}})$*

Proof. If $A \in \text{obj}(\overline{C})$ let $E(A) = y_{\overline{C}}(A)$. If $f \in \overline{C}[D, B], x \in y_{\overline{C}}(A)(B)$ let $E(f)(x) = x \circ f$. Then $E(f) : E(A) \to E(B)$ is a functor. Further, whenever $g : E(A) \to E(B) \in \text{morph}(\overline{(C, J_C)})$, $g(id_A) \in y_{\overline{C}}(B)(A)$ and hence $g(id_A) : A \to B$. But it is clear that $E(g(id_A)) = g$ and so E is surjective on morphisms. If $E(f) = E(g)$ then $f \circ x = g \circ x$ for all $x \in \overline{C}[-, A]$. So, because $(\overline{C}, J_{\overline{C}})$ is subcanonical, and f, g are both covered by $\langle(f \circ x, x) : x \in C[-, A]\rangle$, we have $f = g$. Hence E is injective. So E is an isomorphism of categories. Further it is clear that $S \in J_{\overline{C}}(A)$ if and only if $\widetilde{S} \in J_C(A)$ (where $\widetilde{S}$ is as in Definition 3.13. But as any sieve in $J_C(A)$ is a generating set for $E(A)$ we have $S \in J_{\overline{C}}(A)$ if and only if $E\text{"}[S]$ is an epimorphic family in $\overline{(C, J_C)}$. □

Theorem 3.27. *Suppose (C, J_C) is an almost subcanonical weak sites with $(\overline{C}, J_{\overline{C}})$ the initial element of $Ext(Th_{SubCan}, (C, J_C))$. Then **Sheaf**$(C, J_C)$ is isomorphic to **Sheaf**$(\overline{C}, J_{\overline{C}})$.*

Proof. If $X \in \text{obj}(\mathbf{Sheaf}(\overline{C}, J_{\overline{C}}))$ let $E(X)(A) = X(A)$ for all $A \in \text{obj}(\overline{C}) = \text{obj}(C)$ and if $f : B \to A$ let $E(X)(f) : E(X)(A) \to E(X)(B)$ be the function $f : X(A) \to X(B)$ for all $f \in C[B, A]$. Further if $\alpha \in \mathbf{Sheaf}(\overline{C}, J_{\overline{C}})[X, Y]$ let $E(\alpha)_A = \alpha_A$ for all $A \in \text{obj}(C)$. We then have $E : \mathbf{Sheaf}(\overline{C}, J_{\overline{C}}) \to Sh(C, J_C)$ is a functor.

If $X \in \text{obj}(\mathbf{Sheaf}(C, J_C))$ let $F(X)(A) = X(A)$ for all $A \in \text{obj}(C) = \text{obj}(\overline{C})$ and if $f : B \to A$ let $F(X)(\langle (f_i, i) : i \in T \rangle) : F(X)(A) \to F(X)(B)$ be the function such that for all $b \in F(X)(B)$, $F(X)(\langle (f_i, i) : i \in T \rangle)(a)$ is the unique element covered by $\langle X(f_i)(a) : i \in T \rangle$. Further if $\alpha \in \mathbf{Sheaf}(C, J_C)[X, Y]$ let $F(\alpha)_A = \alpha_A$ for all $A \in \text{obj}(\overline{C})$. We then have $F : \mathbf{Sheaf}(C, J_C) \to \mathbf{Sheaf}(\overline{C}, J_{\overline{C}})$ is a functor.

Further it is immediate that $E \circ F = id_{\mathbf{Sheaf}(C, J_C)}$ and $F \circ E = id_{\mathbf{Sheaf}(\overline{C}, J_{\overline{C}})}$. □

Corollary 3.28. *If (C, J_C) is an almost subcanonical weak site let*

- $(C_0, J_{C_0}) =$ *initial element in* $(Ext(Th_{SubCan}, (C, J_C)))^{V_0}$
- $(C_{\overline{0}}, J_{C_{\overline{0}}}) =$ *initial element in* $(Ext(Th_{SubCan}, (C_0, J_{C_0})))^{V_1}$
- $(C_1, J_{C_1}) =$ *initial element in* $(Ext(Th_{SubCan}, (C, J_C)))^{V_1}$

then $V_1 \models \boldsymbol{Sheaf}(C_0, J_{C_0}) \simeq \boldsymbol{Sheaf}(C_{\overline{0}}, J_{C_{\overline{0}}}) \simeq \boldsymbol{Sheaf}(C_1, J_{C_1})$

Proof. This is because □

Theorem 3.29. *If (C, J_C) and (D, J_D) are equivalent weak sites then $\boldsymbol{Sheaf}(C, J_C)$ and $\boldsymbol{Sheaf}(D, J_D)$ are equivalent categories.*

Proof. Immediate. □

4 Sheaves on a Site

4.1 $Sh(x, \langle C, J_C \rangle)$

In this section we want to explicitly construct a category equivalent to the category of sheaves on a weak site $\langle C, J_C \rangle$ and show that it relativizes. We will then prove some important results concerning this category which will be necessary in order to show that all Grothendieck Topoi relativize.

Definition 4.1. Let $Sh(x, \langle C, J_C \rangle)$ be the formula which says

$$x = \langle Obj, Morph, Dom, Codom, Id, \equiv, \not\equiv \rangle$$

and

(a) $\langle C, J_C \rangle$ is a weak site (here $\langle C, J_C \rangle$ is treated as a parameter).

(b) $Th_{Cat}(x|_{L_{Cat}})$.

(c) $Obj = \{F : C^{op} \to$ **SET** such that F is a separated presheaf for $\langle C, J_C \rangle\}$.

(d i) $Morph = \{\langle D, d, R, r, F \rangle : D, d, R, r \in Sep, F : d \Rightarrow r,$ F is a natural transformation, d, r cover D, R respectively in $(C, J_C^{\mathrm{ORD}})\}$.

(d ii) $\mathrm{dom}(\langle D, d, R, r, F \rangle) = D$ and $\mathrm{codom}(\langle D, d, R, r, F \rangle) = R$. We will often refer to ⟨D, d, R,r, F⟩ simply as F.

(d iii) $\langle X, d, R, r, F \rangle \circ \langle D, d', X, r', G \rangle = \langle D, G^{-1}[r] \cap d', R, r, F \circ G \rangle$

(e) If $X \in Obj$ then $Id(X) = \langle X, X, X, X, id_X \rangle$.

(f i) $(\forall F, G \in Obj) F \equiv G$ if and only if $F = G$.

(f ii) For all $\langle D_F, d_F, R_F, r_F, F \rangle, \langle D_G, d_G, R_G, r_G, G \rangle \in Morph$, $\langle D_F, d_F, R_F, r_F, F \rangle \equiv \langle D_G, d_G, R_G, r_G, G \rangle$ if and only if

- $D_F = D_G$ and $R_F = R_G$
- $(\forall x \in d_F \cap d_G) F(x) = G(x)$

Theorem 4.2. $Set \models (\exists x) Sh(x, \langle C, J_C \rangle)$

Proof. Obj and $Morph$ exist as classes in Set and are uniquely define. As such it suffices to show that the $\equiv$ is an equivalence relation and satisfies conditions of Definition 2.4. First notice that if $\langle D, d_F, R, r_F, F \rangle \equiv \langle D, d_G, R, r_G, G \rangle$ then $\langle \mathbf{a}(D), d_F, \mathbf{a}(R), r_F, F \rangle \equiv \langle D, d_F, R_F, r_F, F \rangle$. Further the unique map from $\mathbf{a}(D)$ to $\mathbf{a}(R)$ induced by F is the same as the map induced by G, so $\langle D_F, d_F, R_F, r_F, F \rangle \equiv \langle D_G, d_G, R_G, r_G, G \rangle$ if and only if the natural transformations they induce from $\mathbf{a}(D)$ to $\mathbf{a}(R)$ are identical.

In particular this means $\equiv$ is an equivalence relation. It is then easy to check the other conditions of Definition 2.4. □

Lemma 4.3. *$Set \models (\forall x, y) Sh(x, \langle C, J_C\rangle) \wedge Sh(y, \langle C, J_C\rangle) \rightarrow x = y$.*

Proof. Because all relations are definable by formulas of set theory which do not mention each other. □

Definition 4.4. If $Set \models Sh(x, \langle C, J_C\rangle)$ then we use $Sh(C, J_C)$ as a shorthand for x.

Theorem 4.5. *$Sh(x, \langle C, J_C\rangle)$ is equivalent to $\boldsymbol{Sheaf}(C, J_C^{\boldsymbol{ORD}})$*

Proof. Let $E(A) = \mathbf{a}(A)$ if A is a separated presheaf. Now if $f = (D, d, R, r, F)$ then there is a unique map $\mathfrak{f} : \mathbf{a}(D) \rightarrow \mathbf{a}(R)$ such that $\mathfrak{f}$ restricted to d is F (this follows from the fact that d covers $\mathbf{a}(D)$ and r covers $\mathbf{a}(R)$). If we let $E(f) = \mathfrak{f}$ then $E : Sh(C, J_C) \rightarrow \mathbf{Sheaf}(C, J_C^{\mathbf{ORD}})$ is a functor. Let $\mathbf{i}$ be the map $\mathbf{Sheaf}(C, J_C^{\mathbf{ORD}}) \rightarrow Sh(C, J_C)$ given by $\mathbf{i}(A) = A$ if $A \in \mathrm{obj}(\mathbf{Sheaf}(C, J_C^{\mathbf{ORD}}))$ and $\mathbf{i}(\alpha) = (\mathrm{dom}(\alpha), \mathrm{dom}(\alpha), \mathrm{codom}(\alpha), \mathrm{codom}(\alpha), \alpha)$ if $\alpha \in \mathrm{morph}(\mathbf{Sheaf}(C, J_C^{\mathbf{ORD}}))$.

If we let $\eta_A : \mathbf{i} \circ E(A) \rightarrow A$ be $(\mathbf{a}(A), A, \mathbf{a}(A), A, id_A)$ then η_A is an isomorphism and η is a natural transformation. Similarly if we let $\varepsilon_A : E \circ \mathbf{i}(A) \rightarrow x$ be id_A then ε_A is an isomorphism and ε is a natural transformation. In particular we have E and $\mathbf{i}$ are equivalences of categories (up to $\equiv$). □

Corollary 4.6. *Suppose $A, B \in obj(Sh(C, J_C))$. Then $(\exists \alpha \in obj(Sh(C, J_C))[A, B])$ α is an isomorphism if and only if the $(\exists \alpha' \in \boldsymbol{Sheaf}(C, J_C^{\boldsymbol{ORD}})[\boldsymbol{a}(A), \boldsymbol{a}(B)])$ α' is an isomorphism .*

For the rest of this paper let $V_0, V_1 \models ST$ be standard models with $V_0 \subseteq V_1$ and let (C, J_C) be an almost subcanonical weak site, in V_0.

Lemma 4.7. *Suppose $(\overline{C}, J_{\overline{C}}) = (C, J_C^{ORD})^{V_0}$. Then $V_1 \models (C, J_C^{ORD}) = (\overline{C}, J_{\overline{C}}^{ORD})$.*

Proof. Because $J_C \subseteq J_{\overline{C}}$, $J_C^{\mathrm{ORD}} \subseteq J_{\overline{C}}^{\mathrm{ORD}} \subseteq (J_C^{\mathrm{ORD}})^{\mathrm{ORD}} = J_C^{\mathrm{ORD}}$. □

Lemma 4.8. *If F is a separated presheaf for (C, J_C) in V_0 then F is a separated presheaf for (C, J_C) in V_1.*

Proof. F is a separated presheaf for (C, J_C) if and only if F is a separated presheaf for $(C, J_C^{\mathrm{ORD}})^{V_0}$ if and only if F is a separated presheaf for $(C, J_C^{\mathrm{ORD}})^{V_1}$. □

Theorem 4.9. *If* $V_0 \models Sh(x, \langle C, J_C\rangle)$ *then* $V_1 \models (\exists y)Sh(y, \langle C, J_C\rangle) \wedge x \subseteq y$

Proof. We see that $Obj^x \subseteq Obj^y$ by Lemma 4.8. Also if F is a separated presheaf then $d \in V_0$ is a covering set for F in V_0 if and only if d is a covering set for F in V_1. Hence $Morph^x \subseteq Morph^y$, $\equiv^x \subseteq \equiv^y$, and $\not\equiv^x \subseteq \not\equiv^y$. So $x \subseteq y$ □

Corollary 4.10. *If* $V_0 \models Sh(x, \langle C, J_C\rangle)$ *then* $V_1 \models Ext(Sh(x, \langle C, J_C\rangle), x)$ *has an initial element object.*

4.2 Limits and Colimits

Theorem 4.11. *Suppose* K *is a diagram in* $Sh(C, J_C)^{V_0}$.

(a) *If* $CoLim \in Sh(C, J_C)^{V_0}$ *is a colimit cocone of* K *in* V_0 *then* $CoLim$ *is a colimit cone of* K *in* $Sh(C, J_C)^{V_1}$

(b) *If* K *is finite and* Lim *is a limit cone of* K *in* $Sh(C, J_C)^{V_0}$ *then* Lim *is a limit cone of* K *in* $Sh(C, J_C)^{V_1}$.

Proof. Part (a):
$CoLim$ is a colimit of K in $Sh(C, J_C)^{V_0}$ if and only if $V_0 \models \mathbf{a}(CoLim)$ is a colimit cone of $\mathbf{a}$"$[K]$ (by Theorem 4.5 and the fact that $\mathbf{a}$ preserves colimits). And similarly $CoLim$ is a colimit of K in $Sh(C, J_C)^{V_1}$ if and only if $V_1 \models \mathbf{a}(CoLim)$ is a limit cone of $\mathbf{a}$"$[K]$. The result then follows from the fact that $V_1 \models \mathbf{a}(\mathbf{a}(CoLim)^{V_0})$ is isomorphic to $\mathbf{a}(CoLim)$.
Part (b):
This is done in an identical way. □

Of course this doesn't mean that if $CoLim \in Sh(C, J_C)^{V_0}$ and $V_1 \models CoLim$ is a colimit of K that we also have $V_0 \models CoLim$ is a colimit of K.

Corollary 4.12. *Suppose* $(G \subset obj(Sh(C, J_C)))^{V_0}$. *If* $V_0 \models$ *"G is a generating set for $Sh(C, J_C)$" then* $V_1 \models$ *"G is a generating set for $Sh(C, J_C)$".*

Proof. We have $y_C(A)$ is a colimit, in V_0, of elements of G. Hence $y_C(A)$ is a colimit of elements of G in V_1. So, as in V_1 all objects of $Sh(C, J_C)$ are colimits of elements of $\{y_C(A) : A \in \text{obj}(C)\}$, in V_1 all objects are colimits of elements of G.

And, because $Sh(C, J_C)$ is a Grothendieck Topos, this implies that G is a generating set for $Sh(C, J_C)$ (in V_1). □

What this corollary shows is that being a generating set is upwards absolute. However, it is not in general downwards absolute.

4.3 Generating Sets

Definition 4.13. Let $G \subseteq \text{obj}(Sh(C, J_C))$. Define $(\widehat{G}, J_{\widehat{G}})$ to be the site where

- $\widehat{G}$ is the full subcategory of $Sh(C, J_C)$ with objects G.
- $S \in J_G(A)$ if and only if S is an epimorphic family in $Sh(C, J_C)$.

Theorem 4.14. *Let $V_0 \models$ "(C, J_C) is an almost subcanonical weak site" and let $\mathcal{C} = \{y_C(A) : A \in obj(C)\}$. If $V_0 \models \mathcal{C} \subseteq D \subseteq obj(Sh(C, J_C))$ then $V_1 \models Sh((\widehat{\mathcal{C}}, J_{\widehat{\mathcal{C}}})^{V_0})$ is equivalent to $Sh((\widehat{D}, J_{\widehat{D}})^{V_0})$*

Proof. First notice by Corollary 3.28 $V_1 \models Sh(C, J_C) \simeq Sh((\widehat{C}, J_{\widehat{C}})^{V_0})$. To simplify notation we will use (C, J_C) instead of $(\widehat{\mathcal{C}}, J_{\widehat{\mathcal{C}}})^{V_0}$. Also notice that $V_1 \models (C, J_C)$ and $(\widehat{D}, J_{\widehat{D}})^{V_0}$ are almost subcanonical by Theorem 2.31. Working in V_1 let $G = Sh(C, J_C)$ and let $H = Sh((\widehat{D}, J_{\widehat{D}})^{V_0})$.

In this proof we will have the same separated presheaves occurring in several different categories. As such it is worth fixing some notation to deal with this. If $d \in D \subseteq \text{obj}(Sh(C, J_C)^{V_0})$ let d_0 be d treated as an element of $\text{obj}(Sh(C, J_C)^{V_0})$ and let d_1 be d treated as an element of $\text{obj}(Sh(C, J_C)^{V_1}) = G$. Further let $d_0^* = y_{\overline{D}}(d)$ be the image of d in $Sh(\overline{D}, J_{\overline{D}})^{V_0}$ and $d_1^* = y_{\overline{D}}(d))$ be the image of d in $Sh((\overline{D}, J_D)^{V_0})^{V_1} = H$. Likewise we will let $D_G = \{d_1 : d \in D\}$ and $D_H = \{d_1^* : d \in D\}$

Now by Theorem 4.5 $V_1 \models Sh(C, J_C)$ is equivalent to $Sh(\widehat{\mathcal{C}}, J_{\widehat{\mathcal{C}}})$ which is equivalent to $Sh(\widehat{D_G}, J_{\widehat{D_G}})$ (because $\mathcal{C}$ is a generating set for $Sh(\overline{C}, J_{\overline{C}})$ so is D_G). Also by Corollary 4.12 then $V_1 \models H$ is equivalent to $Sh(\widehat{D_H}, J_{\widehat{D_H}})$. So in particular to show that $V_1 \models Sh(\widehat{D_H}, J_{\widehat{D_H}})$ is equivalent to $Sh(\widehat{D_G}, J_{\widehat{D_G}})$ it suffices to show H is equivalent to G. Or it suffices to show $V_1 \models (\exists E)E : (\widehat{D_G}, J_{\widehat{D_G}}) \to (\widehat{D_H}, J_{\widehat{D_H}})$ where E is an isomorphism of sites.

We will construct E explicitly. First notice that there are injective maps $F_G : (\widehat{D}, J_{\widehat{D}})^{V_0} \to (\widehat{D_G}, J_{\widehat{D_G}})$ and $F_H : (\widehat{D}, J_{\widehat{D}})^{V_0} \to (\widehat{D_H}, J_{\widehat{D_H}})$ which are isomorphisms on objects. The map F_H coming from the fact that, by Corollary 3.28 $(\widehat{D_H}, J_{\widehat{D_H}})^{V_1}$ is an initial object in $Ext(Th_{SubCan}, (\widehat{D}, J_{\widehat{D}})^{V_0})$. The map F_G coming from the fact that $Sh(C, J_C)^{V_0} \subseteq Sh(C, J_C)^{V_1}$ (by Theorem 4.9

). Lastly notice that these maps are maps of sites and not just categories as any cover in $(\widehat{D}, J_{\widehat{D}})^{V_0}$ is also a cover in the other categories, $(\widehat{D_G}, J_{\widehat{D_G}})$ and $(\widehat{D_H}, J_{\widehat{D_H}})$, under these maps

If $X \in \text{obj}(\widehat{D_G})$ let $E(X) = F_H(F_G^{-1}(X))$ and if $Y \in \text{obj}(\widehat{D_H})$ let $E'(Y) = F_G(F_H^{-1}(Y))$. Then E, E' are inverses functions on the objects of $\widehat{D_G}, \widehat{D_H}$ respectively because F_G, F_H are bijections on objects.

For each $A \in \text{obj}(\widehat{D})$ let $Cov_G(A) = \{F_G(f) : f \in C[X, A], X \in \text{obj}(C)\}$ and let $Cov_H(A) = \{F_H(f) : f \in C[X, A], X \in \text{obj}(C)\}$. Then $Cov_G(A)$ is a covering set for $F_G(A)$ because $\{f : f \in C[X, A], X \in obj(C)\}$ is a covering set of A and $F_G(A)$ is a sheafification of A. Similarly $Cov_H(A)$ is a covering set of $F_H(A)$ because $\{y_{\widehat{D}}(f) : f \in C[X, A], X \in \text{obj}(C)\}$ is a covering set of $y_{\widehat{D}}(A)$ and $F_H(A)$ is a sheafification of $y_{\widehat{D}}(A)$. Further $(\forall A \in \text{obj}(\widehat{D}))(\exists$ isomorphism $i : Cov_G(A) \simeq Cov_H(A)$ such that $i(f) = y_{\widehat{D}}(f)$.

If $\alpha \in \widehat{D_G}[F_G(A), F_G(B)]$ let $E(\alpha) \in \widehat{D_H}[E(F_G(A)), E(F_G(B))]$ be the unique map such that $E(\alpha)(i(f)) = i(\alpha(f))$ for all $f \in Cov_G(A)$. Similarly if $\alpha' \in \widehat{D_H}[F_H(A), F_H(B)]$ let $E'(\alpha') \in \widehat{D_H}[E'(F_H(A)), E'(F_H(B))]$ be the unique map such that $E'(\alpha')(i^{-1}(f)) = i^{-1}(\alpha(f))$ for all $f \in Cov_H(A)$. It is then clear that E, E' are functors and inverses of each other. Hence E is an isomorphism.

All that is left is to show that $J_{\widehat{D_H}} = J_{\widehat{D_G}}$. However we know that $S \in J_{D_G}(A)$ if and only if the map $\phi_S : \coprod\{y_{D_G}(\text{dom}(f)) : f \in S\} \to A$ is an epimorphism if and only if there are covering sets $X_i \subseteq Cov(y_{D_G}(\text{dom}(f)))$ and $X_A \subseteq Cov(A)$ such that if X is the coproduct of the the X_i's then X_A is in the image of X. But this holds if and only if the same holds once E is applied □

Theorem 4.15. *Suppose (C, J_C) and (D, J_D) are almost subcanonical weak sites such that $V_0 \models Sh(C, J_C)$ is equivalent to $Sh(D, J_D)$. Then $V_1 \models Sh(C, J_C)$ is equivalent to $Sh(D, J_D)$*

Proof. First notice by Corollary 3.28 we can assume without loss of generality that (C, J_C) and (D, J_D) are subcanonical sites in V_0 (and hence almost subcanonical in V_1).

By assumption there are maps, such that $V_0 \models E_C : Sh(C, J_C) \to Sh(D, J_D)$ and $E_D : Sh(D, J_D) \to Sh(C, J_C)$ are equivalences of categories. Let $(\mathcal{C}, J_{\mathcal{C}})$ be the full subcategory of $Sh(C, J_C)^{V_0}$ whose objects are $\{y_C(A) : A \in \text{obj}(C)\} \cup \{E_D(y_C(B)) : B \in \text{obj}(D)\}$ and whose covering sieves are epimorphic families. Similarly let $(\mathcal{D}, J_{\mathcal{D}})$ be the full subcategory of $Sh(D, J_D)^{V_0}$

whose objects are $\{y_D(A) : A \in \text{obj}(D)\} \cup \{E_C(y_D(B)) : B \in \text{obj}(C)\}$ and whose covering sieves are epimorphic families. Because E_C, E_D are equivalences of categories, they restrict an equivalences of sites between $(\mathcal{C}, J_{\mathcal{C}})$ and $(\mathcal{D}, J_{\mathcal{D}})$.

Hence by Theorem 3.29 we have $V_1 \models Sh(\mathcal{C}, J_{\mathcal{C}}) \simeq Sh(\mathcal{D}, J_{\mathcal{D}})$. But we also have by Theorem 4.14 that $V_1 \models Sh(C, J_C) \simeq Sh(\mathcal{C}, J_{\mathcal{C}})$ and $Sh(D, J_D) \simeq (\mathcal{D}, J_{\mathcal{D}})$. □

5 Grothendieck Toposes

Unlike in the case of sheaves on a site, for a general Grothendieck Topos G as we change models of set theory the category of expansions of G which are Grothendieck Topoi has more than one object. In this section we will give a theory whose models are exactly (a definable expansion) of the Grothendieck Topoi such that every model has a relativizes to every standard model of set theory.

Definition 5.1. Let G be a Grothendieck Topos. We say that (C, J_C) is a *generating site* for G if

- $C \subseteq G$ and C is a set.
- J_C consists of epimorphic families in C.
- G is equivalent to $Sh(C, J_C)$

Definition 5.2. Let $L_{Topoi} = L_{Cat} \cup \{GS\}$ where GS is unary relations on the power set of the model. Let $GT(x)$ be the formula which says

- $x|_{L_{Cat}}$ has a set of generators.
- Sums in $x|_{L_{Cat}}$ are disjoint.
- All equivalence relations in $x|_{L_{Cat}}$ are effective.
- $GS(g)$ if and only if $g \subseteq x$ is a generating site for x.

The first three of these conditions are *Giraud's axioms for a Grothendieck Topos* ([4]).

Theorem 5.3. *Suppose $V_0 \models GT(G)$. Then $(obj(Ext(GT,G)))^{V_1}$ is non-empty.*

Proof. We know that $V_0 \models (\exists(C,J_C))E_C : G \simeq Sh(C,J_C)$. Working in V_1 let $I : \mathrm{obj}(G) \cup \mathrm{obj}(Sh(C,J_C)) \to \mathrm{obj}(Sh(C,J_C))$ where $I|_{Sh(C,J_C)}$ is the identity and $I|_G$ is E_C. We then let

- $\mathrm{obj}(G') = \mathrm{obj}(G) \cup \mathrm{obj}(Sh(C,J_C))$
- $G'[X,Y] = \{(x,y,f) : f \in Sh(C,J_C), f : I(x) \to I(y)\}$ where $(x,y,f) \circ (y,z,g) = (x,z,g \circ f)$.

G' is then clearly equivalent to G and there is a full and faithful injection $I' : G \rightarrowtail G'$ such that $I'|_{\mathrm{obj}(G)} = id_G$ and if $f \in G[X,Y]$ then $I'(f) = (X,Y,f)$.

Now G' isn't an extension of G because we don't have G as a subset of G'. But if we define $H = (G' - image(I')) \cup G$ with the obvious composition of morphisms then H is an extension of G (with respect to L_{Cat}) and there is an isomorphism between H and G'.

So all that is left is to show that $(H \models GS(g,J_g))^{V_1}$ whenever $(G \models GS(g,J_g))^{V_0}$. Now $(G \models GS(g,J_g))^{V_0}$ if and only if $(G \simeq Sh(g,J_g))^{V_0}$. But by Theorem 4.15 we have $(H \simeq Sh(C,J_C) \simeq Sh(g,J_g))^{V_1}$ because $V_0 \models Sh(C,J_C) \simeq Sh(g,J_g)$. Hence $(H \models GS(g,J_g))^{V_1}$. So $(\mathrm{obj}(Ext(GT,G)))^{V_1}$ is non-empty. □

Theorem 5.4 **(*).** *If $V_1 \models$ Axiom of Choice then $(Ext(GT,G))^{V_1}$ has an initial element (and hence G has a relativization).*

Proof. Working in V_1 let $Skel(C,J_C)$ be a skeletal subcategory of $Sh(C,J_C)$ (which we know exists as V_1 satisfies the axiom of choice). Let H' be the full subcategory of H (from Theorem 5.3 whose objects are either from G or are from $Skel(C,J_C)$ and not isomorphic to any objects in G. H' is then equivalent to H which is equivalent to $Sh(C,J_C)$.

As H' is equivalent to H, any generating site in H also is a generating set in H' (under the equivalence) and so the inclusion map from G into H' preserves generating sites. So, $(H' \in Ext(GT,G))^{V_1}$.

By Theorem 4.15 if $(T \in Ext(GT,G))^{V_1}$ then T is equivalent to $Sh(C,J_C)$ (in V_1). But if T is any such category containing G as a subcategory, then there must be a unique injection from H' into T taking G to itself. Hence H' is an initial element of $(Ext(GT,G)^{V_1})$. □

We will end with a conjecture

Conjecture 5.1. *If $GT^*(x)$ is the L_{Cat} formula which says, x satisfies Giraud's axioms for a Grothendieck Topos then all models of GT^* have a relativization to all standard models of set theory with the Axiom of Choice. Further the relativization of a model of GT is isomorphic (in L_{Cat}) to its relativization as a model of GT^*.*

In other words if we remove the condition that our Grothendieck Topos preserve generating sites then we still have that all Grothendieck Topoi relativize and they all relativize to the same categories (i.e. if $G \simeq Sh(C, J_C)$ in V_0 then $G' \simeq Sh(C, J_C)$ in V_1 (where G' is the relativization)).

6 *

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