Γ-Ultrametric Spaces and Presheaves

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Abstract

In this paper we will explore the relationship between Γ-Ultrametric Spaces and Separated Presheaves. We will then use this relationship to prove some new results about Separated Presheaves with nice properties.

Γ-Ultrametric spaces (as in this paper) have been extensively studied (see for example [7], [8], [1]) as a way to weaken the notion of an ultrametric space while still providing enough structure to be useful. Γ-Ultrametric spaces are spaces which satisfy all the axioms of an ultrametric space except that the distance function takes values in a fixed complete lattice (and not necessarily in the reals) [1]

We begin this paper in Section 1 by reviewing the relevant definitions before discussing in Section 2 the close connection between the categories of Γ-Ultrametric Spaces and the category of separated presheaves on Γ^{op}.

In Section 3, we will show how all categories with epi-monic factorizations can be viewed as having Hom-sets which are Γ-Ultrametric spaces. This will then allow us, in Section 4, to translate results about Generalized Ultrametric Spaces (like the Priess-Crampe/Ribenboim Fixed Point Theorem ([6]) into results about objects in other categories.

1 Background

In this section we will review some of the categorical definitions we will need. For more information on category theory the reader is referred to such standard texts as [4]. For more information on sheaf theoretic ideas the

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[1] There are several other ways in which one can generalize the notion of Ultrametric Space. For more information on some of these see notions see [3] or [9].

reader is referred to such standard works such as [5]. For more information on generalized ultrametric space (as defined here) the reader is referred to [7].

All categories in this paper will be locally small and we will use the convention that when C is a category with objects A and B, $C[A,B]$ is the set of morphisms whose domain is A and whose codomain is B (i.e. a Hom set). We will also use $C[-,B]$ for the set of all morphisms whose codomain is B.

Throughout this paper we will assume there is a model, Set, of Zermelo-Frankel Set Theory in the background (see [2]). **SET** will be the category whose objects are sets in this model and whose morphisms are functions between sets in this model. We will not assume that this model satisfies the Axiom of Choice. However, some of our arguments will require the Axiom of Choice and these arguments will be marked with a (*) before their statement.

1.1 Γ-Ultrametric Spaces

Definition 1.1. Let $(\Gamma, \leq, 0)$ be a complete lattice with minimal element 0 [2] A Γ-*ultrametric space* is a pair (M, d_M) such that

- M is a set and $d_M : M \times M \to \Gamma$.
- (Reflexivity) $(\forall x, y \in M) d_M(x,y) = 0 \leftrightarrow x = y$
- (Symmetry) $(\forall x, y \in M) d_M(x,y) = d_M(y,x)$
- (Strong Triangle Inequality) $(\forall x, y, z \in M) d_M(x,y) \vee d_M(y,z) \geq d_M(x,z)$

A *non-expanding* map between Γ-ultrametric spaces (M, d_M) and (N, d_N) is a function $f : M \to N$ such that $(\forall a, b \in M) d(f(a), f(b)) \leq d(a,b)$. This gives us a category **Γ-UltMet** whose objects are Γ-ultrametric spaces and whose morphisms are the non-expanding maps.

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[2] If we weaken (Strong Triangle Inequality) to say

$$(\forall x, y, z \in M)(\forall \gamma \in \Gamma) d_M(x,y) \leq \gamma \text{ and } d_M(y,z) \leq \gamma \text{ then } d_M(x,z) \leq \gamma$$

Then this notion make sense for any partial order $(\Gamma, \leq, 0)$ with a least element. However, as we will never care if there are some elements of Γ which aren't realized as distances we can assume without loss of generality in this paper that Γ is a complete lattice.

For the rest of Section 1 we will let $(\Gamma, \leq, 0)$ be a complete lattice and we will let (M, d_M) be a Γ-ultrametric space.

Definition 1.2. For each $x \in M, \gamma \in \Gamma$ we define the *(closed) ball of radius* γ *around* x to be $B^M(x, \gamma) = \{y : d_M(x, y) \leq \gamma\}$. We will omit mention of M when it is clear from context.

The following lemmas are immediate and the proofs are left to the reader.

Lemma 1.3. $(\forall \gamma \in \Gamma, x, y \in M) x \in B(y, \gamma) \leftrightarrow B(x, \gamma) = B(y, \gamma)$

Lemma 1.4. $(\forall \gamma_x, \gamma_y \in \Gamma)(\forall x, y \in M) \gamma_x \leq \gamma_y \rightarrow B(x, \gamma_x) \subseteq B(y, \gamma_y)$ *or* $B(x, \gamma_x) \cap B(y, \gamma_y) = \emptyset$.

Definition 1.5. If $A \subseteq M$ the *diameter of* A is $\operatorname{diam}(A) = \bigvee\{d_M(x, y) : x, y \in M\}$.

Theorem 1.6. *Suppose*

- $\alpha = \bigwedge\{\gamma_i : i \in I\}$ *and* $\{x_i : i \in I\} \subseteq M$
- $B = \bigcap_{i \in I} B(x_i, \gamma_i) \neq \emptyset$

Then $(\forall x \in B) B = B(x, \alpha)$.

Proof. Suppose $x \in B$. Then $(\forall i \in I) x \in B(x_i, \gamma_i)$ and so $B(x, \gamma_i) = B(x_i, \gamma_i)$. Hence $(\forall i \in I) B \subseteq B(x, \gamma_i)$ and $\operatorname{diam}(B) \leq \gamma_i$. Therefore $\operatorname{diam}(B) \leq \alpha$ and in particular $B \subseteq B(x, \alpha)$. But we also have $(\forall i \in I) B(x, \alpha) \subseteq B(x, \gamma_i)$ and so $B(x, \alpha) \subseteq B$. $\square$

While much of our focus will be on Γ-ultrametric spaces for a specific Γ, it will at times be useful to be able to compare generalized ultrametric spaces with different sets of distances.

Definition 1.7. If (M, d_M) is a Γ-ultrametric space and (N, d_n) is an Λ-ultrametric space we say a map $f : M \rightarrow N$ is *relative distance preserving* if $(\forall a, b, c, d \in M) d_M(a, b) \leq d_M(c, d) \rightarrow d_N(f(a), f(b)) \leq d_N(f(c), f(d))$. We then let **GenUltMet** be the category whose objects are Γ-ultrametric spaces for some Γ and whose morphisms are the relative distance preserving maps.

1.2 Category Theory

Definition 1.8. Let C be a category and let $A \in \text{obj}(C)$. The collection of subobjects of A is the set $Sub_C(A) = \{m : X \to A, m \text{ is monic}\}/\equiv$ where $m \equiv m'$ if there is an isomorphism i such that $m = m' \circ i$. We let $[m]$ be the subobject containing m. We say $[m] \leq [m']$ if there is a monic map i with $m = m' \circ i$.

Definition 1.9. Let C be a category, $A \in \text{obj}(C)$, $m : X \rightarrowtail A$ and $f : B \to A$. Let $f^{-1}([m])$ be the subobject of B which is the pullback of m along f.

Definition 1.10. We say that a category C has *epi-mono factorization* if for all morphisms $f \in C[A, B]$ there are maps $f_e : A \twoheadrightarrow im(f)$ and $f_m : im(f) \rightarrowtail B$ such that $f = f_m \circ f_e$, f_e is an epimorphism and f_m is a monomorphism. Further if $f = f'_m \circ f'_e$ where f_m is a monomorphism and f_e is an epimorphism then there is an isomorphism i such that $f'_m = f_m \circ i$ and $f'_e = i^{-1} \circ f_e$.

1.3 Presheaves and Sheaves

Definition 1.11. If C, D are categories we say C is *isomorphic* to D ($C \cong D$) if there are functors $i_C : C \to D$ and $i_D : D \to C$ such that $i_C \circ i_D = id_D$ and $i_D \circ i_C = id_C$. We say C is *equivalent* to D ($C \simeq D$) if there are functors $e_C : C \to D$ and $e_D : D \to C$ as well as natural isomorphisms $\eta_C : e_D \circ e_C \Rightarrow 1_C$ and $\eta_D : e_C \circ e_D \Rightarrow 1_D$.

Definition 1.12. Let C be a category and $F : C^{op} \to \mathbf{SET}$ a presheaf on C. An *element of* F is an element of $\bigcup_{A \in \text{obj}(C)} F(A)$. If $f \in C[A, B], x \in F(B)$ we will use the shorthand $x|_f$ for $F(f)(x)$.

Definition 1.13. Let C be a category and $A \in \text{obj}(C)$. A *sieve* S on A is a subfunctor of $C[-, A]$. If S is a sieve on A and $f : B \to A$ then the *pullback of* S *along* f is the sieve on B given by $f^*S(D) = \{g \in C[D, B] \text{ such that } f \circ g \in S\}$.

Definition 1.14. A *site* is a pair (C, J_C) where C is a category and J_C a function from the objects of C to collections of sieve such that for any $A \in \text{obj}(C)$:

- (Identity) $C[-, A] \in J_C(A)$

- (Base Change) If $S \in J_C(A)$ and $f : B \to A$ then $f^*S \in J_C(B)$
- (Local Character) Let $S \in J_C(A)$ and let T be any sieve on A. If $(\forall f \in S(B)) f^*T \in J_C(B)$ then $T \in J_C(A)$.

We call $J_C(A)$ the *covering sieves of A*

Definition 1.15. If $(\Gamma, \leq, 0)$ is a complete lattice we will view it as a site where $(\Gamma, \leq)$ is treated as a category and $S \in J_\Gamma(X)$ if and only if $\bigvee S = X$. If $F : \Gamma^{op} \to \mathbf{SET}$ is a presheaf, $a \in F(X)$ and $X \geq Y$ we will use $a|_Y$ as a shorthand for $F(i_{Y,X})(a)$ (where $i_{Y,X}$ is the unique map from Y to X).

Definition 1.16. Let (C, J_C) be a weak site and $F : C^{op} \to \mathbf{SET}$ be a presheaf on C. If $A \in \mathrm{obj}(C)$, a *compatible collection of elements on A* is a collection $\langle (a_i, i) : i \in S \rangle$ such that

- $S \in J_C(A)$.
- $(\forall i \in S(B)) a_i \in F(B)$
- $(\forall i' : B' \to B) a_{i \circ i'} = F(i')(a_i)$

If there is an $a \in F(A)$ such that $a|_i = a_i$ for all $i \in S$ then we say a is *covered by* $\langle (a_i, i) : i \in X \rangle$.

Definition 1.17. Suppose (C, J_C) is a weak site. A presheaf $F : C^{op} \to \mathbf{SET}$ is *separated for* (C, J_C) if for every compatible collection of elements $\langle (a_i, i) : i \in S \rangle$ on A there is a most one $a \in F(A)$ covered by $\langle (a_i, i) : i \in S \rangle$. We let $\mathbf{Sep}(C, J_C)$ be the category whose objects are separated presheaves for (C, J_C) and whose morphisms are the natural transformations.

Definition 1.18. Suppose (C, J_C) is a weak site. A presheaf $F : C^{op} \to \mathbf{SET}$ is a *sheaf for* (C, J_C) if for every compatible collection of elements $\langle (a_i, i) : i \in S \rangle$ on A there is exactly one $a \in F(A)$ covered by $\langle (a_i, i) : i \in S \rangle$. We let $\mathbf{Sheaf}(C, J_C)$ be the category whose objects are sheaves for (C, J_C) and whose morphisms are the natural transformations.

2 Properties of Γ-Ultrametric Spaces and Presheaves

2.1 Spherically Complete

An important property that a Γ-ultrametric spaces may have is what is called spherical completeness. This is the Γ-ultrametric analog of completeness for metric spaces.

Definition 2.1. A Γ-ultrametric space (M, d_M) is *spherically complete* if whenever

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ and $\{x_i : i < \kappa\} \subseteq M$
- $B(\gamma_i, x_i) \subseteq B(\gamma_j, x_j)$ if $i \geq j$.

Then $\bigcap_{i<\kappa} B(\gamma_i, x_i) \neq \emptyset$. We define **SComp(Γ)** to be the full subcategory of **Γ-UltMet** whose objects are spherically complete.

Theorem 2.2. *Suppose (M, d_M) is a spherically complete Γ-ultrametric space and*

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ *and* $\{x_i : i < \kappa\} \subseteq M$
- $B(\gamma_i, x_i) \subseteq B(\gamma_j, x_j)$ *if* $i \geq j$.

is a decreasing chain of closed balls. Then $(\forall x \in \bigcap_{i<\kappa} B(\gamma_i, x_i)) B(\bigwedge_{i<\kappa} \gamma_i, x) = \bigcap_{i<\kappa} B(\gamma_i, x_i)$

Proof. This is an immediate consequence of Theorem 1.6. □

2.2 Pruned Presheaves

Definition 2.3. We say a separated presheaf A on Ω is *pruned* if $(\forall \gamma \in \Omega^{op})(\forall a \in A(\gamma))(\exists a' \in A(1)) a'|_\gamma = a$. [3] We define **Pruned(Ω)** to be the full subcategory of **Sep**(Ω) whose objects are the pruned presheaves. Likewise we define **PrunedSheaf(Ω)** to be the full subcategory of **Sheaf(Ω)** whose objects are pruned presheaves.

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[3] If Ω is the lattice of open sets on ω (with the initial segment topology) then a separated presheaf on Ω is simply a tree. A separated presheaf on Ω is pruned (as in Definition 2.3) if and only if it is pruned as a tree (and this is why the presheaves in Definition 2.3 are called *pruned*).

The main result of this section is that pruned presheaves on Ω and Ω^{op}-ultrametric spaces are essentially the the same. In other words

Theorem 2.4. *There is an equivalence of categories between* ***Pruned(*** Ω ***)*** *and* Ω^{op}***-UltMet****.*

Proof. In order to be consistent the names of all elements will be as in Ω. We will use a superscript ^{op} to signify the corresponding notion in Ω^{op}. For example $a \leq b$ if and only if $a \geq^{op} b$ and $1 = 0^{op}$.

Claim 2.5. *If* (A, d_A) *is an* Ω^{op} *ultrametric space let* $A^*(\gamma) = \{B(a,\gamma) : a \in A\}$ *where* $B(a,\gamma)|_{\gamma^*} = B(a,\gamma^*)$ *for all* $\gamma^* \leq \gamma$*. Then* A^* *is a separated pruned presheaf on* Ω

Proof. First we need to confirm that restriction is well defined. Suppose $B(a,\gamma) = B(b,\gamma) \in A^*(\gamma)$, $\gamma^* \leq \gamma$ and $x \in B(a,\gamma^*)$ (i.e. $d_A(x,a) \leq^{op} \gamma^*$). $d_A(a,b) \leq^{op} \gamma \leq^{op} \gamma^*$, so $d_A(x,b) \leq^{op} \gamma^*$ and $x \in B(b,\gamma^*)$. Hence $B(a,\gamma^*) = B(b,\gamma^*)$ and restriction is well defined.

To see that A^* is pruned notice that if $B(a,\gamma) \in A^*(\gamma)$ then $B(a,0^{op}) \in A^*(1)$ and $B(a,0^{op})|_\gamma = B(a,\gamma)$.

Finally, to see A^* is separated, suppose $\gamma = \bigvee_{i\in I}\lambda_i$ and $\boldsymbol{B} = \{B(x_i,\lambda_i) \in A^*(\lambda_i) : i \in I\}$ is such that $B(x_i,\lambda_i)|_{\lambda_i\wedge\lambda_j} = B(x_i,\lambda_i\wedge\lambda_j) = B(x_j,\lambda_i\wedge\lambda_j) = B(x_j,\lambda_j)|_{\lambda_i\wedge\lambda_j}$. Now if $\bigcap_{i\in I} B(x_i,\lambda_i) = \emptyset$ then there is no element in $A^*(\gamma)$ compatible with $\boldsymbol{B}$ as such an element would have be of the form $B(x,\gamma)$ with $B(x_i,\lambda_i) = B(x,\lambda_i)$ for all $i \in I$ and hence $x \in \bigcap_{i\in I} B(x_i,\lambda_i) = \emptyset$. On the other hand, if there exists $x \in \bigcap_{i\in I} B(x_i,\lambda_i)$ then by Theorem 1.6 we have $B(x,\gamma) = \bigcap_{i\in I} B(x_i,\lambda_i)$ and hence $B(x,\gamma)$ is the unique element of $A^*(\gamma)$ compatible with $\mathcal{B}$. □

Claim 2.6. *If* $(A, d_A), (C, d_C) \in$ *obj(*Ω^{op}***-UltMet****) and* $f : (A, d_A) \to (C, d_C)$ *is a non-expanding map, let* $f^*(B^A(a,\gamma)) = B^C(f(a),\gamma)$*. Then* $f^* \in$ (***Pruned(*** Ω ***)***)$[A^*, C^*]$

Proof. First we need to show that $f^*(B(a,\gamma))$ doesn't depend on our choice of a. Suppose $B^A(a,\gamma) = B^A(b,\gamma)$ and $x \in B^C(f(a),\gamma)$. Then $d_C(f(a),f(b)) \leq^{op} d_A(a,b) \leq \gamma$ and $d_C(x,f(a)) \leq^{op} \gamma$. So $d_C(x,f(b)) \leq^{op} \gamma$ and $x \in B^C(f(b),\gamma)$. Hence $f^*(B^A(a,\gamma)) = f^*(B^A(b,\gamma))$.

Next we need to show f^* is a natural transformation, i.e. that if $\lambda \geq \gamma$ then $f^*_\lambda(B^A(a,\gamma)|_\lambda) = f^*_\gamma(B^A(a,\gamma))|_\lambda$. But $f^*_\lambda(B^A(a,\gamma)|_\lambda) = f^*_\lambda(B^A(a,\lambda)) = B^C(f(a),\lambda) = B^C(f(a),\gamma)|_\lambda = f^*_\gamma(B^A(a,\gamma))|_\lambda$. So f^* is a natural transformation from A^* to C^*. □

Let F be the functor where $F(A, d_A) = A^*$ if $(A, d_A) \in \text{obj}(\Omega^{op}\textbf{-UltMet})$ and $F(f) = f^*$ if $f \in \text{morph}(\Omega^{op}\textbf{-UltMet})$.

Claim 2.7. *If A is pruned separated presheaf on Ω let (A^o, d_A) be such that $A^o = A(1)$ and $d_A(a, b) = \bigvee\{\gamma : a|_\gamma = b|_\gamma$. Then (A^o, d_A) is an Ω^{op}-ultrametric space.*

Proof. First notice $a|_{d_A(a,b)} = b|_{d_A(a,b)}$ because $\{a|_\gamma : \gamma \leq d_A(a, b)\}$ is a compatible collection of elements covering both $a|_{d_A(a,b)}$ and $b|_{d_A(a,b)}$. In particular this means that if $d_A(a, b) = 0^{op}$ then $a = a|_0 = b|_0 = b$. So (A^o, d_A) satisfies (reflexivity). Also, (symmetry) is immediate from the definition. To see the (strong triangle inequality) holds let $a, b, c \in A(1)$ with $d_A(a, b) \leq^{op} \gamma$ and $d_A(b, c) \leq^{op} \gamma$. Then $a|_\gamma = b|_\gamma = c|_\gamma$ and hence $d_A(a, c) \leq^{op} \gamma$. □

Claim 2.8. *If $A, C \in$ obj(**Pruned**(Ω)) and $f \in$ **Pruned**$(\Omega))[A, C]$ then $f_1 : A(1) \to C(1)$ is a non-expanding map $f_1 : (A^o, d_A) \to (C^o, d_C)$.*

Proof. We know that for all $a, b \in A(1)$, $a|_{d_A(a,b)} = a|_{d_A(a,b)}$ and so $f_1(a)|_{d_A(a,b)} = f_{d_A(a,b)}(a|_{d_A(a,b)}) = f_{d_A(a,b)}(b|_{d_A(a,b)}) = f_1(b)|_{d_A(a,b)}$ and hence $d_A(a, b) \geq^{op} d_C(f_1(a), f_1(b))$. □

Let E be the functor where $E(A) = (A^o, d_A)$ if $A \in \text{obj}(\textbf{Pruned}(\Omega))$ and $E(f) = f_1$ if $f \in \text{morph}(\textbf{Pruned}(\Omega))$

Claim 2.9. *For all $A \in$ obj(**Pruned**(Ω)) there is an isomorphism $\eta_A : A \Rightarrow F(E(A))$ which is the identity on $A(1)$.*

Proof. First notice that for all $a, b \in A(1)$, $a|_\gamma = b|_\gamma$ if and only if $d_A(a, b) \leq^{op} \gamma$ if and only if $B^{A^o}(a, \gamma) = B^{A^o}(b, \gamma)$. So the maps $(\eta_A)_\gamma(a|_\gamma) = B^{A^o}(a, \gamma)$ is well defined and an injective natural transformation. Further, because A is pruned, $(\eta_A)_\gamma$ is also surjective and hence an isomorphism for all γ. So η_A is a natural isomorphism. □

Claim 2.10. *For any map $f \in$ **Pruned**$(\Omega)[A, C]$ we have $\eta_C \circ f = (f_1)^* \circ \eta_A$.*

Proof. As all four maps are maps of pruned presheaves they are determined by their values on $A(1)$. □

Hence $\eta : 1_{\textbf{Pruned}(\Omega)} \Rightarrow F \circ E$ is a natural isomorphism.

Claim 2.11. *For all $(A, d_A) \in$ obj(Ω^{op}**-UltMet**) there is a natural isomorphism $\varepsilon_A : (A, d_A) \Rightarrow E(F(A, d_A))$.*

Proof. If $(A', d_{A'}) = E(F(A, d_A))$ then $A' = \{\{a\} : a \in A\}$ and for all $a, b \in A$, $d_{A'}(\{a\}, \{b\}) = \bigwedge^{op}\{\gamma : B^A(a, \gamma) = B^A(b, \gamma)\} = d_A(a, b)$. Hence the map $\varepsilon_A(a) = \{a\}$ is an isomorphism of Ω^{op}-ultrametric spaces. □

Claim 2.12. *For any map $f \in \Omega^{op}$-**UltMet**$[(A, d_A), (C, d_C)]$ we have $\varepsilon_C \circ f = (f^*)_1 \circ \varepsilon_A$.*

Proof. Immediate. □

Hence $\varepsilon : 1_{\Omega^{op}\textbf{-UltMet}} \Rightarrow F \circ E$ is a natural isomorphism. And in particular E, F are equivalences of categories. □

Theorem 2.13 (*)**.** *The equivalence of Theorem 2.4 restricts to an equivalence of categories between **PrunedSheaf(Ω)** and **SComp(Ω^op)**.*

Proof.

Claim 2.14. *If (A, d_A) is a spherically complete Ω^{op} ultrametric space then $F(A) = A^*$ is a sheaf.*

Proof. Let $I = \langle \gamma_j : j < \kappa \rangle \subseteq \Omega^{op}$ with $\langle (a_i, i) : i \in I \rangle$ be a compatible collection of elements of A^* (with I a sieve). So there are x_i such that $a_i = B(x_i, \gamma_i)$. We define B_i by induction.

- $B_0 = B(x_0, \gamma_0)$.
- $B_{\alpha+1} = B(x_{\gamma_{\alpha+1}}, \gamma_{\alpha+1}) \cap B_\alpha$
- $B_{\omega\cdot\beta} = \bigcap_{j<\omega\cdot\beta} B_j$.

We now want to show $(\forall l < \kappa, \lambda \leq \kappa) B(x_{\gamma_l}, \gamma_l) \cap B_\lambda \neq \emptyset$. To get a contradiction assume j is least such that $B(x_{\gamma_l}, \gamma_l) \cap B_j = \emptyset$ for some $l < \kappa$. We break into three cases.

Case 1: $j = 0$
This case can't happen because $\langle (a_i, i) : i \in I \rangle$ is a compatible collection of elemetns and hence the a_i's are closed under intersection.

Case 2: $j = \omega \cdot \beta$.
Notice that $B_r \subseteq B_s$ if $s \leq r < \omega \cdot \beta$. So $B(x_{\gamma_l}, \gamma_l) \cap B_{\omega\cdot\beta} = B(x_{\gamma_l}, \gamma_l) \cap \bigcap_{h<\omega\cdot\beta} B_h = \bigcap_{h<\omega\cdot\beta}(B(x_{\gamma_l}, \gamma_l) \cap B_h)$. But $B(x_{\gamma_l}, \gamma_l) \cap B_h \neq \emptyset$ by the inductive hypothesis and $\langle B(x_{\gamma_l}, \gamma_l) \cap B_h : h < \omega \cdot \beta \rangle$ is a decreasing sequence of balls.

Hence $\bigcap_{h<\omega\cdot\beta} B(x_{\gamma_l},\gamma_l)\cap B_h \neq \emptyset$ because (A,d_A) is spherically complete. So this case can't happen.

Case 3: $j=\alpha+1$.
$B(x_{\gamma_l},\gamma_l)\cap B_{\alpha+1} = B(x_{\gamma_l},\gamma_l)\cap(B(x_{\gamma_\alpha},\gamma_\alpha)\cap B_{\alpha+1}) = B(x_{\gamma_l\wedge\gamma_\alpha},\gamma_l\wedge\gamma_\alpha)\cap B_\alpha \neq \emptyset$. So this case can't happen and we have our contradiction.

We therefore have $B_\kappa = \bigcap_{i<\kappa} B_i \neq \emptyset$ and so, by Theorem 1.6, $B_\kappa = B(x,\bigvee I)$ for any $x\in B_\kappa$. Hence B_κ is the unique element of $A^*(\bigvee I)$ which is covered by $\langle (a_i,i) : i\in I\rangle$. So, because $\langle (a_i,i) : i\in I\rangle$ was arbitrary, A^* is a sheaf. □

Claim 2.15. *If A is a pruned sheaf on Ω then (A^o,d_A) is a spherically complete Ω^{op}-ultrametric space.*

Proof. Suppose

- $\{\gamma_i : i<\kappa\}\subseteq\Omega^{op}$ and $\{x_i : i<\kappa\}\subseteq A$
- $B^{A^o}(\gamma_i,x_i)\subseteq B^{A^o}(\gamma_j,x_j)$ if $i\geq j$.

Whenever $j\leq i$ we then have $x_i\in B^{A^o}(\gamma_j,x_j)$ and $\gamma_i\leq\gamma_j$ and so $x_j|_{\gamma_i} = x_i|_{\gamma_i}$. Hence $(\forall i,j<\kappa)x_j|_{\gamma_i\wedge\gamma_j} = x_i|_{\gamma_i\wedge\gamma_j}$ and $\langle (x_i|_\zeta : i<\kappa,\zeta\leq\gamma_i\rangle$ is a compatible collection of elements. But because A is a sheaf there is an $y\in A(\bigvee_{i<\kappa}\gamma_i)$ covered by $\langle (x_i|_\zeta : i<\kappa,\zeta\leq\gamma_i\rangle$. Further, because A is a pruned sheaf we know there is an $x^*\in A(1)=A^o$ such that $x^*|_{\bigvee_{i<\kappa}\gamma_i} = y$. So for all $i,j<\kappa$, $x^*|_{\gamma_i} = x_i|_{\gamma_i}$ hence $x^*\in B^{A^o}(\gamma_i,x_i)$ and $x^*\in\bigcap_{i<\kappa}B^{A^o}(\gamma_i,x_i)\neq\emptyset$. So (A^o,d_A) is spherically complete (as our sequence of balls was arbitrary). □

□

As we will see in Section 3, this equivalence between **Pruned**(Ω) and Ω^{op}-**UltMet** comes from the fact that in the category of separated presheaves on Ω the set of maps between any two objects has a Ω^{op}-ultrametric structure. The pruned presheaves are then exactly those objects which are the colimit of a diagram all of whose objects are 1, with the corresponding Ω^{op}-ultrametric space being $Hom(1,A)$.

This equivalence will allow us to generalize several well known results concerning Γ-Ultrametric Spaces to results about separated presheaves and sheaves.

2.3 Properties of Γ-UltMet

Now that we have a way to describe the category of Γ-ultrametric spaces it is natural to ask what properties this category has. As we will see in this section the category **Γ-UltMet** is complete, cocomplete and (assuming the Axiom of Choice) has exponentials.

Theorem 2.16. *The inclusion map* $i : \textbf{Pruned}(\Gamma) \to \textbf{Sep}(\Gamma)$ *has a right adjoint* r.

Proof. If $A \in \mathrm{obj}(\textbf{Sep}(\Gamma))$, let $r(A)(\gamma) = \{x \in A : (\exists y \in A(1)) y|_\gamma = x\}$. If $f \in \textbf{Sep}(\Gamma)[A, B]$ then we know that the image of f restricted to $r(A)$ is determined by f_1. As such if we let $r(f)_X(x) = f_X(x)$ for all $x \in r(A)(X)$ then $r(f)_X(x) \in r(B)_X$ for all $x \in r(A)(X)$. Hence we have r is a functor.

Let $\eta_X = id_X : X \to r \circ i(X)$ and let $\varepsilon_X : i \circ r(X) \to X$ be the inclusion map. Then it is clear that $\eta : 1_{\textbf{Sep}(\Gamma)} \Rightarrow r \circ i$ and $\varepsilon : i \circ r \Rightarrow 1_{\textbf{Sep}(\Gamma)}$ are the unit and counit (respectively) of an adjunction. $\square$

Corollary 2.17. ***Pruned*(Γ)** *is complete.*

Proof. Because $r \circ i \cong 1_{\textbf{Pruned}(\Gamma)}$, $i \dashv r$ and $\textbf{Sep}(\Gamma)$ is complete. $\square$

Notice though that r does not restrict to an adjunction for the inclusion map $i : \textbf{PrunedSheaf}(\Gamma) \to \textbf{Sheaf}(\Gamma)$ because even if A is a sheaf, there is no guarantee that $r(A)$ will be as well. In general, **PrunedSheaf(Γ)** is not a complete category because it does not necessarily have all equalizers.

Theorem 2.18 (*)**.** *If* A *is a pruned separated presheaf on* Ω *and* B *is any separated presheaf then* A^B *is a pruned separated presheaf.*

Proof. We have for each γ, $A^B(\gamma) \cong A(\gamma)^{B(\gamma)}$ (because we are only dealing with localic topoi up to this point). Suppose $f \in A^B(\gamma)$ and for each $x \in A(\gamma)$, $g(x) \in A(1)$ and $g(x)|_\gamma = x$ (we know such a g exists as A is pruned and we can choose once because we have assumed the Axiom of Choice).

Let $\langle x_i : i < \kappa \rangle = A(\gamma)$. Define A_i as follows. $A_0 = \{x_0|_{\gamma'} : \gamma' \leq \gamma\}$, $A_\alpha = \{x_\alpha|_{\gamma'} : \gamma' \leq \gamma\} - \bigcup_{i<\alpha} A_i$. If $y \in A_i$ let $\overline{g}(y) = g(x_i)$. So $\overline{g}$ takes an element of $a \in A(\eta)$ $(\eta \leq \gamma)$ and returns a $\overline{g}(a) \in A(1)$ such that $\overline{g}(a)|_\eta = a$. So in particular $\overline{g}(a|_\zeta) = \overline{g}(a)|_\zeta$.

Next let $f^* \in A^B(1)$ be define so that if $b \in B(\lambda)$ then $[f^*|_\lambda](b) = \overline{g}([f|_{\lambda\wedge\gamma}](b|_{\lambda\wedge\gamma}))|_\lambda$. We then have, by the previous paragraph, that $[f^*|_\lambda](b)|_\eta =$

$\overline{g}([f|_{\lambda\wedge\gamma}](b|_{\lambda\wedge\gamma}))|_\eta = \overline{g}([f|_{\eta\wedge\gamma}](b|_{\eta\wedge\gamma}))|_\eta = [f^*|_\eta](b|_\eta)$. Hence f^* yields a map of presheaves.

So $[f^*] \in A^B(1)$ and $f^*|_\gamma = f$ and hence A^B is pruned. $\square$

Theorem 2.19. ***Pruned(*** Γ***)*** *is cocomplete.*

Proof. It suffices to show that the colimit of pruned presheaves (in the category of presheaves) is a pruned presheaf. But this follows immediately from the fact that in the category of presheaves colimits are taken pointwise. $\square$

3 Hom Sets and Generalized Ultrametric Spaces

In this section we will show that in many categories the Hom sets (i.e. sets of the form $C[A,B]$ where $A,B \in \mathrm{obj}(C)$) can be viewed as generalized ultrametric spaces.

3.1 General

Definition 3.1. Let C be a category with $G \subseteq \mathrm{obj}(C)$ and let $P(G,A) = Powerset(\bigcup_{X\in G} C[X,A])$. For every $A,B \in \mathrm{obj}(C)$ define $d_G : C[A,B] \times C[A,B] \to P(G,A)$ as follows:

- $d_G(f,g) = \{h : X \to A \text{ such that } f \circ h = g \circ h\}$.

For every $k : B \to D$ we define a map $k_! : (C[A,B], d_G) \to (C[A,D], d_G)$ by $k_!(g) = k \circ g$

Theorem 3.2. *If G is a generating set of objects for C then $(C[A,B], d_G)$ is a $P(G,A)$-ultrametric space (where $\alpha \le \beta$ if and only if $\beta \subseteq \alpha$ and $0 = \bigcup_{X\in G} C[X,A]$). Further if $f : B \to D$ then each map $f_!$ is non-expanding.*

Proof. First lets show that $(C[A,B], d_G)$ is a $P(G,A)$-ultrametric space. (Symmetry) is immediate. For (Reflexivity) notice that because G is a generating set of objects if $f,g \in C[A,B]$ and $f \neq g$ then there is some $a : X \to A$ with $X \in G$ such that $f \circ a \neq g \circ a$. In particular if $f \neq g$ then $d_G(f,g) \neq 0$.

To show the strong triangle inequality suppose $f,g,h \in C[A,B]$. Whenever $a : D \to A$ where $a \in d_G(f,g) \cap d_G(g,h)$ we have $f \circ a = g \circ a = h \circ a$ and hence $a \in d_G(f,h)$. Therefore $d_G(f,g) \cap d_G(g,h) \subseteq d_G(f,h)$ or equivalently $d_G(f,g) \vee d_G(g,h) \ge d_G(f,h)$.

To see that any $k_!$ is a non-expanding map notice that if $a \in d_G(f,g)$ then $f \circ a = g \circ a$ and hence $k \circ f \circ a = k \circ g \circ a$ and $a \in d(k \circ f, k \circ g) = d(k_!(f), k_!(g))$. So $d(k_!(f), k_!(g)) \supseteq d(f,g)$ or equivalently $d(k_!(f), k_!(g)) \leq d(f,g)$.

□

Corollary 3.3. *If C is a category and G is a generating set then for each $A \in obj(C)$ there are functor $F_{G,A} : C \to P(G,A)$-**UltMet** given by*

- $(\forall B \in obj(C)) F_{G,A}(B) = (C[A,B], d_G)$
- $(\forall k \in C[B,D]) F_{G,A}(k) = k_!$

Proof. This follows from the fact that $k_! \circ j_! = (k \circ j)_!$. □

Lemma 3.4. *$F_{G,A}$ preserves products that exist in C.*

Proof. Notice that the product $\Pi_{i \in I}(C[A,B_i], d_{G_i}) = (\Pi_{i \in I} C[A,B] \times C[A,D], d'_G)$ where $d'_G((a_i : i \in I), (b_i : i \in I)) = \bigwedge_{i \in I} d_{G_i}(a_i, b_i)$. The result then follows from the fact that $C[A, \Pi_{i \in I} B_i] \cong \Pi_{i \in I} C[A, B_i]$ and for any two maps $f, g : A \to \Pi_{i \in I} B_i$ and any map $x : X \to A$, $x \circ f = x \circ g$ if and only if $(\forall i \in I) \pi_i \circ x \circ f = \pi_i \circ x \circ g$ (where π_i is the projection onto B_i). □

3.2 Lattice Of Subobjects

If we know that our categories have epi-mono factorizations then, up to isomorphism in **GenUltMet**, our choice of generating set doesn't matter in determining the generalized ultrametric space structure put on Hom sets of the category.

For the rest of Section 3 let C be category with epi-mono factorization.

Definition 3.5. Suppose C is a category with epi-mono factorizations. Let $P(Sub_C) = Powerset(Sub_C(A))$. For every $A, B \in \text{obj}(C)$ define $d_S : C[A,B] \times C[A,B] \to P(S)$ as $d_S(f,g) = \{[h] : f \circ h = g \circ h\}$.

For every $k : B \to D$ we define a map $k_! : (C[A,B], d_S) \to (C[A,D], d_S)$ by $k_!(g) = k \circ g$

Theorem 3.6. *If $G \subseteq obj(C)$ then for each $A, B \in obj(C)$, and $f, g, x, y \in C[A,B]$ we have*

(a) $d_S(x,y) \leq d_S(f,g) \Rightarrow d_G(x,y) \leq d_G(f,g)$.

(b) If G is a generating set for C then $d_G(x,y) \leq d_G(f,g) \Rightarrow d_S(x,y) \leq d_S(f,g)$.

Proof. Part (a):
Suppose $a : D \rightarrow A$ with $D \in G$. a then has an epi-mono factorization $a = a_m \circ a_e$. Now $a \in d_G(x,y)$ if and only if $x \circ (a_m \circ a_e) = y \circ (a_m \circ a_e)$ if and only if $x \circ a_m = y \circ a_m$ if and only if $[a_m] \in d_S(x,y)$ (the second to last equivalence follows because a_e is an epimorphism). So if $[a_m] \in d_S(x,y) \rightarrow [a_m] \in d_S(f,g)$ then $a \in d_G(x,y) \rightarrow a \in d_G(f,g)$. Hence $d_S(x,y) \leq d_S(f,g)$ implies $d_G(x,y) \leq d_G(f,g)$.

Part (b):
To get a contradiction suppose

- G is a generating set for C
- $d_G(x,y) \leq d_G(f,g)$
- $a : D \rightarrowtail A$ with $[a] \in d_S(f,g)$
- $[a] \notin d_S(x,y)$

Then $x \circ a \neq y \circ a$ and hence there must be a map $c : C \rightarrow B$ such that $x \circ a \circ c \neq y \circ a \circ c$ and $c \in G$. So $a \circ c \notin d_G(x,y)$ and hence by assumption $a \circ c \notin d_G(f,g)$ and $f \circ a \circ c \neq g \circ a \circ c$. But $f \circ a = g \circ a$ (by definition of $[a] \in d_S(f,g)$) and so $f \circ a \circ c = g \circ a \circ c \Rightarrow\Leftarrow$. □

Corollary 3.7. *If G is a generating set for C and $A, B \in obj(C)$ then $(C[A,B], d_S)$ is isomorphic to $(C[A,B], d_G)$ in* ***GenUltMet****.*

Corollary 3.8. *If C is a category then for each $A \in obj(C)$ there is a functor $F_{Sub,A} : C \rightarrow P(Sub, A)$-**UltMet** given by*

- $(\forall B \in obj(C)) F_{Sub,A}(B) = (C[A,B], d_S)$
- $(\forall k \in C[B,D]) F_{Sub,A}(k) = k_!$

Further $F_{Sub,A}$ preserves all products which exist in C.

Proof. This follows from the fact that $k_! \circ j_! = (k \circ j)_!$. □

Theorem 3.9. *Let $F_{Sub,1}$ be as in Corollary 3.8, E be as in Theorem 2.4, and r as in Theorem 2.16. Then there exists $\eta : E \circ r \Rightarrow F_{Sub,1}$ a natural isomorphism.*

Proof. This follows immediately from the definition of $E \circ r$ and F along with the fact that $P(1) \cong \Omega$ as a lattice. □

4 Translation of Results on Γ-Ultrametric Spaces

Now that we have characterized the category of Γ-ultrametric spaces in terms of presheaves on Γ^{op} and shown how the Hom sets of $\mathbf{Sep(}\Gamma^{op}\mathbf{)}$ can be viewed as Γ-ultrametric spaces, we want to use these facts to translate results about Γ-ultrametric spaces into results about pruned presheaves.

In this section we will provide necessary and sufficient conditions for various properties of ultrametric spaces to hold on the space of maps between two pruned presheaves. We will then give an example of how this can be used to translate results about generalized ultrametric spaces into results about pruned presheaves. Specifically we will show how these translations allow us to prove a generalization of the Priess-Crampe/Ribenboim Fixed Point Theorem (which is itself a generalization of the Banach Fixed Point Theorem).

4.1 Complete and Pruned

For Section 4 let C be a category with epi-mono factorization.

Definition 4.1. We say that an object A is *D-pruned* if $(\forall e : E \rightarrowtail D)(\forall f : E \to A)(\exists h : D \to A)$ such that $h \circ e = f$ (where e is a monic).

This is a generalization of the notion of a pruned presheaf. Specifically a presheaf is pruned if and only if it is 1-pruned.

Definition 4.2. Suppose C has all colimits and $A, B \in \text{obj}(C)$. We say B is *A-complete* if

- For all complete lattices $(\Gamma, \leq)$ and all sequences $\langle e_i : E_i \rightarrowtail A : i \in \Gamma\rangle$ with $e_{i,j} : E_i \to E_j$ such that $e_j \circ e_{i,j} = e_i$ whenever $i \leq j$
- For all sequences functions $\langle f_i \in C[A, B] : i \in \Gamma\rangle$ with $f_i \circ e_j = f_j \circ e_j$ whenever $j \leq i$.

There is a map $f^* : A \to B$ with $f^* \circ e_i = f_i \circ e_i$ for all $i \in \Gamma$.

Theorem 4.3 (*). *Suppose C has all colimits and $A, B \in obj(C)$. Then $(C[A, B], d_S)$ is spherically complete if and only if B is A-complete.*

Proof. $\Rightarrow$:
Let $\Gamma, \langle e_i : E_i \rightarrowtail A : i \in \Gamma\rangle, \langle f_i \in C[A,B] : i \in \Gamma\rangle$ be as in the definition. Further let $\langle \alpha_s : s < \kappa\rangle \subseteq \Gamma$ be an increasing sequence of elements of Γ such that $\bigvee\langle \alpha_s : s < \kappa\rangle = \bigvee \Gamma$. Let $B_s = \{g \in C[A,B] : g \circ e_{\alpha_s} = f_{\alpha_s}\}$. Then B_s is a closed ball and $B_s \subseteq B_t$ if $s \geq t$. So, by spherical completeness, there exists $f^* \in \bigcap_{s<\kappa} B_s$

$\Leftarrow$:
For each $i < \kappa$ let $B_i = B(f_i, E_i)$ where $E_i \subseteq Sub_C(A)$ and if $B(f_i, E_i) = B(f_i, E')$ then $E' \subseteq E_i$. Further suppose $B_i \subseteq B_j$ if $i \geq j$ and so $E_j \subseteq E_i$ if $j \leq i$. Now $B_i = B(f_i, Colimit(E_i))$ and if $e_i \in Colimit(E_i)$ then $\langle f_i : i < \kappa\rangle$ and $\langle e_i : i < \kappa\rangle$ satisfy the conditions of the theorem. Hence there must be a f_κ such that $B(f_\kappa, Colimit(E_i)) = B_i$ and so $f_\kappa \in \bigcap_{i<\kappa} B_i$. Therefore $C[A,B]$ is spherically complete. □

Theorem 4.4. *Suppose $F, F', G, G' : B \to D$ are maps in C and suppose $x, y \in C[A,B]$. Then the statement $d_S(F(x), G(y)) \leq d_S(F'(x'), G'(y'))$ holds if and only if $(\forall z : X \rightarrowtail A) G' \circ y' \circ z = F' \circ x' \circ z \to G \circ y \circ z = F \circ x \circ z$*

Proof. This follows immediately from the definitions. □

4.2 Contracting Maps

For the rest of Section 4 fix a generating set G of objects of C.

Definition 4.5. We say a map $a : A \to B$ in C is *contracting for D* if $(\forall f, g \in C[D,A]) d_S(a(f), a(g)) < d_S(f,g)$

Theorem 4.6. *A map $a : A \to B$ is contracting for D if and only if $(\forall f, g : D \to A)(\exists h : E \to D)$ such that $f \circ h \neq g \circ h$ but $a \circ f \circ h = a \circ f \circ h$. We call h a witness that a is contracting for f and g.*

Proof. This follows immediately from the fact that $d_S(a(f), a(g)) < d_S(f,g)$ if and only if $d_G(a(f), a(g)) < d_G(f,g)$ (Theorem 3.6) □

Theorem 4.7. *If $a : A \to B$ is contracting for D then $(\forall f, g : D \to A)(\exists h : E \to D)$ such that $f \circ h \neq g \circ h$ but $a \circ f \circ h = a \circ f \circ h$ with $E \in G$.*

Proof. Suppose $f, g : D \to A$ and $h : E' \to D$ are such that $f \circ h \neq g \circ h$ but $a \circ f \circ h = a \circ g \circ h$. Then there is a map $i : E \to E'$ such that $E \in X$ and $f \circ h \circ i \neq g \circ h \circ i$. Hence $h \circ i : E \to D$ is the desired witness. □

Theorem 4.8. *If $a : A \to B$ is contracting for all $D \in G$ then a is contracting for all $E \in obj(C)$.*

Proof. Suppose $f, g : E \to A$ are such that $f \neq g$. Then there is a $Y \in X$ and $h : Y \to E$ such that $f \circ h \neq g \circ h$. But by assumption a is contracting for Y and so there is a $k : Z \to Y$ such that $f \circ h \circ k \neq g \circ h \circ k$ but $a \circ f \circ h \circ k = a \circ f \circ h \circ k$. Hence $h \circ k$ witnesses that a is contracting for E (with respect to f, g). ☐

Theorem 4.9. *If $a : A \to B$ is a contracting map for D and $b : B \to C$ is any map, then $b \circ a : A \to C$ is a contracting map for D.*

Proof. Suppose $f, g : D \to A$ are such that $f \neq g$. Then there is a $k : Z \to D$ such that $f \circ k \neq g \circ k$ but $a \circ f \circ k = a \circ g \circ k$. But then we also have $b \circ a \circ f \circ k = b \circ a \circ g \circ k$. Hence $b \circ a$ is contracting for D. ☐

Theorem 4.10. *If $a : A \to B$ is a contracting map for D and $c : C \to A$ is any map, then $a \circ c$ is a contracting map for D.*

Proof. Suppose $f, g : D \to C$. We then have two cases. In the first case $c \circ f = c \circ g$ and hence $a \circ c \circ f = a \circ c \circ g$. But then id_D witnesses $a \circ c$ is contracting for f, g. In the second case we have $c \circ f \neq c \circ g$. However both of these are maps from D to A and so there must be a $k : E \to D$ such that $(c \circ f) \circ k \neq (c \circ g) \circ k$ and $a \circ (c \circ f) \circ k = a \circ (c \circ g) \circ k$. Hence k witnesses that $c \circ a$ contracts f, g. ☐

Notice though that if $f : A \to B$ is a contracting map of pruned presheaves in **Sep(Ω)** that doesn't mean that f is a contracting map in the category **Pruned(Ω)**. In general in **Pruned(Ω)** there are no contracting maps (other than those from the 1).

4.3 Fixed Point Theorem

For this section fix a Grothendieck Topos G. Given any map $a : A \to A$ define $Fix(a)$ to be subobject of A corresponding to the equalizer of a and 1_A.

Theorem 4.11 (*). *Suppose $a : A \to A$ is a contracting map for D and A is D-pruned. Then there is a unique map $f : D \to A$ such that $a \circ f = a$. We call f the D-fixed of a.*

Proof. First notice that because $0 : 0 \rightarrowtail A$ and $[0] \in Sub_G(D)$ we know there is a map $h : D \to A$. We will prove the existence of such a map f by repeated applying the contracting map a. Define $f_\alpha : D \to A$ as follows

- $f_0 = h$, and $Fix_0 = f_0^{-1}[im(f_0) \wedge Fix(a)]$ (which is a subobject of D)
- $f_{\alpha+1} = a \circ f_\alpha$ and $Fix_{\alpha+1} = f^{-1}[im(f_{\alpha+1}) \wedge Fix(a)]$ (which is a subobject of D)
- Let $Fix_{\omega\cdot\gamma} = \bigvee_{i<\omega\cdot\gamma} Fix_i$ and let $f^*_{\omega\cdot\gamma} : \bigvee_{i<\omega\cdot\gamma} f_i^{-1}[im(f_i) \cap Fix(a)] \to A$ be a map agreeing with f_i on Fix_i (for all $i < \omega \cdot \gamma$). Then define $f_{\omega\cdot\gamma}$ to be any map from D into A which factors through $f^*_{\omega\cdot\gamma}$ (which we know exists as A is D-pruned). [4]

Notice that $Fix_\alpha \subseteq Fix_{\alpha+1}$ by construction and if $Fix_\alpha \neq id_D$ then by Theorem 3.6 and the fact that a is contracting for D, we have $Fix_\alpha \subsetneq Fix_{\alpha+1}$. So for some α, $f_\alpha : D \to A$ is a fixed point.

Now suppose there are two $f, g : D \to A$ such that $a \circ f = f$ and $a \circ g = g$. Then $d_S(a(f), a(g)) = d_S(a \circ f, a \circ g) = d_S(f, g)$. Hence, because a is contracting for D, we must have $f = g$. So there is a unique fixed point of a. □

Because A is 1-pruned if and only if A is pruned this theorem reduces to the Priess-Crampe/Ribenboim Fixed Point Theorem ([6]) in the case $D = 1$.

5 *

References

[1] Ulrich Heckmanns. *Aspects of ultrametric spaces*, volume 109 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University.

[2] Thomas Jech. *Set Theory: The Third Millennium Edition.* Springer Monogrphas in Mathematics. Springer-Verlag, 2000.

[3] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 1973.

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[4] This case works because G is a Grothendieck Topos and hence the lattice of subobjects of A is a complete lattice.

[4] Saunders Mac Lane. *Categories for the Working Mathmatician*. Graduate Texts in Mathematics. Springer-Verlag, 2nd edition.

[5] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. Springer-Verlag, 1st edition.

[6] Sibylla Priess-Crampe and Paulo Ribenboim. Fixed points, combs and generalized power series. *Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg*, 1993.

[7] Sibylla Priess-Crampe and Paulo Ribenboim. Generalized ultrametric spaces. i. *Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg*, 1996.

[8] Sibylla Priess-Crampe and Paulo Ribenboim. Generalized ultrametric spaces. ii. *Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg*, 1997.

[9] Lynn Arthur Steen and J. Arthur Seebach Jr. *Counterexamples in Topology*. Springer-Verlag, 1978.