Math 412: Final Exam

All questions are worth 10 points. There is no extra credit. Not all questions are necessarily the same difficulty, nor does difficulty necessarily increase along with problem number. You are allowed FIS (any edition), your notes, your old HW, and old HW solutions. You are also allowed a calculator, although it should not be necessary to have one in order to complete the exam. You may use the calculator only for NUMERICAL functions (i.e., no symbolic calculations of characteristic polynomials or stuff like that—an unjustified symbolic calculation that seems as if it was done on the calculator will lose points). If you have a question about what this constitutes, let me know. When in doubt, justify your assertions. Good luck!

1. Say that $A \in M_{n\times n}(\mathbb{C})$ is skew-symmetric, that is, $A^T = -A$. Prove that if n is odd, then A is not invertible (*Hint:* Determinants).

2. Say we are in $\mathbb{R}^3$ with the standard inner product (i.e., dot product). Find the orthogonal projection of the vector $(-3, 4, 6)$ onto the plane W determined by the condition $2x+y+z=0$. Also, find a basis for $W^\perp$.

3. (FIS p. 298, #11) In 1940 a county land-use survey showed that 10% of the county land was urban, 50% was unused, and 40% was agricultural. Five years later, a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentage of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages. (*Hint:* Show that the transition matrix is regular).

4. (FIS p.354, #6b) Let T be a linear transformation from $V \to V$, with V an inner product space. Let W be a T-invariant subspace of V. Prove that $W^\perp$ is T^*-invariant.

5. Let $A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$.

a) Find the characteristic polynomial of A, and for each eigenvalue, find a basis of the eigenspace. Then find Q such that $Q^{-1}AQ$ is a diagonal matrix D. Before actually writing down Q, how do you know that A is diagonalizable?

b) It is not hard to show that, for any invertible $n\times n$ matrix P, and any $n \times n$ matrices M and N, we have $P^{-1}MP + P^{-1}NP = P^{-1}(M+N)P$ (you do not need to prove this). With this in mind, and noting that $A = QDQ^{-1}$, verify, without using the calculator, that A satisfies the Cayley-Hamilton theorem (you can do this directly, but your calculation will be easier if you can figure out how to use the above hints).