

## Final Exam Solutions

1. Since  $A^T = -A$ , we have  $\det A = \det A^T = \det(-A) = (-1)^n \det A$ . Since  $n$  is odd, we have  $\det A = -\det A$ . So  $\det A = 0$ , which means that  $A$  is not invertible.

2. We need to find an orthonormal basis for  $W$ . We start by finding any basis for  $W$  and then we apply Gram-Schmidt. A basis is given by  $w_1 = (0, 1, -1), w_2 = (1, 0, -2)$ . Applying Gram-Schmidt, we first normalize  $w_1$  to get  $v_1 = \frac{1}{\sqrt{2}}(0, 1, -1)$ . Then, we calculate  $w_2 - \langle w_2, v_1 \rangle v_1$  which is  $(1, 0, -2) - \frac{2}{\sqrt{2}}(\frac{1}{\sqrt{2}}(0, 1, -1))$ . This comes out to  $(1, -1, -1)$ . Normalizing this gives  $v_2 = \frac{1}{\sqrt{3}}(1, -1, -1)$ .

Now, to find the orthogonal projection of  $v = (-3, 4, 6)$ , we use the formula  $\langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2$ . This gives  $(0, -1, 1) + (-\frac{13}{3}, \frac{13}{3}, \frac{13}{3})$ . Adding these together gives that the orthogonal projection of  $v$  to  $W$  is  $v' = (-\frac{13}{3}, \frac{10}{3}, \frac{16}{3})$ .

Lastly, we know that  $v - v' \in W^\perp$ .  $v - v' = (\frac{4}{3}, \frac{2}{3}, \frac{2}{3})$ . Since  $W^\perp$  is one dimensional, any nonzero vector forms a basis. So a basis consists of the lone vector  $(\frac{4}{3}, \frac{2}{3}, \frac{2}{3})$ .

3. The transition matrix for the given problem is  $A = \begin{pmatrix} .7 & .2 & 0 \\ .1 & .6 & .2 \\ .2 & .2 & .8 \end{pmatrix}$ , and the initial state

vector is  $x = \begin{pmatrix} .1 \\ .5 \\ .4 \end{pmatrix}$ , where the first coordinate represents urban land, the second coordinate represents unused land, and the third coordinate represents agricultural land. Then, the land distribution use in 1950 (which is two time periods after the beginning) is given by

$A^2 x = \begin{pmatrix} .197 \\ .339 \\ .464 \end{pmatrix}$ . So 19.7% of the land is urban, 33.9% of the land is unused, and 46.4% of the land is agricultural in 1950.

To calculate the eventual percentages, we note that  $A^2 = \begin{pmatrix} .51 & .26 & .04 \\ .17 & .42 & .28 \\ .32 & .32 & .68 \end{pmatrix}$  has all positive

entries, so  $A$  is regular. So there is a unique equilibrium, given by the fixed probability vector. So we only need to calculate the eigenspace corresponding to 1. Solving  $Av = v$  is the same thing

as solving  $(A - I)v = 0$ . A solution to this equation is  $v = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$  (there are no other linearly

independent solutions). Normalizing, so that  $v$  is a probability vector, gives  $\begin{pmatrix} .2 \\ .3 \\ .5 \end{pmatrix}$ . So eventu-

ally, 20% of the land will be urban, 30% will be unused, and 50% of the land will be agricultural.

4. Let  $w' \in W^\perp$ . In order to show that  $W^\perp$  is  $T^*$ -invariant, we only need to show that  $T^*(w') \in W^\perp$ , that is, that  $\langle T^*(w'), w \rangle = 0$  for all  $w \in W$ . But  $\langle T^*(w'), w \rangle = \langle w', T(w) \rangle$ . Since  $W$  is  $T$ -invariant, we have that  $T(w) \in W$ . Since  $w' \in W^\perp$  and  $T(w) \in W$ , then by definition we have that  $\langle w', T(w) \rangle = 0$ . So  $\langle T^*(w'), w \rangle = 0$  and we are done.

5. a) The characteristic polynomial of  $A$  is given by

$$\det \begin{pmatrix} 3-t & -1 & 1 \\ 0 & 2-t & 0 \\ 1 & 1 & 3-t \end{pmatrix}.$$

Expanding via the second row gives  $(2-t)((3-t)^2 - 1) = (2-t)(t^2 - 6t + 8) = (2-t)^2(4-t)$ . So the eigenvalues are 2 and 4. The space  $E_2$  is the kernel of left multiplication by  $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ ,

a basis for which is given by  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$  (we know that it is two dimensional because the rank of the matrix  $A - 2I$  is 1). The space  $E_4$  is the kernel of left multiplication by  $\begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$ , a basis for which is given by  $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$  (we know that it is one dimensional because the rank of the matrix  $A - 4I$  is 2). So we set

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

to get that  $Q^{-1}AQ = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ . We know that  $A$  has to be diagonalizable, because its characteristic polynomial splits, and  $m_2 = \dim E_2 = 2$  and  $m_4 = \dim E_4 = 1$ .

b) Expanding out the characteristic polynomial of  $A$  gives  $-t^3 + 8t^2 - 20t + 16$ , so we want to show that  $-A^3 + 8A^2 - 20A + 16I = 0$ . Since  $A = QDQ^{-1}$ , this is equivalent to showing that

$$-(QDQ^{-1})^3 + 8(QDQ^{-1})^2 - 20(QDQ^{-1}) + 16I = 0.$$

But using the observation from the statement of the problem (and the fact that  $QIQ^{-1} = I$ ), we can rewrite this as  $Q(-D^3 + 8D^2 - 20D + 16I)Q^{-1} \stackrel{?}{=} 0$ , or as

$$Q \begin{pmatrix} -2^3 + 8 \cdot 2^2 - 20 \cdot 2 + 16 & 0 & 0 \\ 0 & -2^3 + 8 \cdot 2^2 - 20 \cdot 2 + 16 & 0 \\ 0 & 0 & -4^3 + 8 \cdot 4^2 - 20 \cdot 4 + 16 \end{pmatrix} Q^{-1} \stackrel{?}{=} 0.$$

But this is clearly true, as it is equal to  $Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$ , which is equal to 0.