Take-Home Midterm
Due Tuesday, July 19th

Each problem is worth 10 points, for a total of 80. There are also 5 points of extra credit available in problem 4. Although all problems are worth the same, I do not expect them all to be of equal difficulty, nor does difficulty necessarily increase along with problem number (the order of the problems roughly corresponds to the order in which we covered the material). So if a problem is causing you trouble, let it rest for a while and move on to the next one. You may use the book, your class notes, and the posted previous homework solutions, but you may not work together. If any questions require clarification, please let me know. Show your work!

Many of these are taken from FIS or from the practice problem sheet, but in most cases I have modified either the problem or the notation, and in some cases I have added a hint.

1. For each of the following subsets of $\mathbb{R}^3$, indicate whether or not it is a subspace. If it is not a subspace, justify why not (if it is a subspace, you don't need to prove it).

- a) $\{(0, y, 0) | y \in \mathbb{R}\}$.
- b) $\{(1, y, 0) | y \in \mathbb{R}\}$.
- c) $\{(x, y, x + y) | x, y \in \mathbb{R}\}$.
- d) $\{(x, y, x + 1) | x, y \in \mathbb{R}\}$.
- e) $\{(x, x^2, x^3) | x \in \mathbb{R}\}$.
- f) $\{(x, y, z) \in \mathbb{R}^3 | x + 2y - z = 0\}$.
- g) $\{(x, y, z) \in \mathbb{R}^3 | x - y = 2y + 3 = z - 1 = 0\}$.
- h) $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 0\}$.
- i) $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$.
- j) $\{(x + y, -z, u + t) \in \mathbb{R}^3 | x - u = z + t = 2y - t = 0\}$.

2. Give bases for the following vector spaces. What are their dimensions?

- a) $\{M \in M_{2\times2}(\mathbb{R}) | M \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot M\}$
- b) $\{f \in P_3(\mathbb{R}) | f(0) = f''(0)\}$

3. Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times k}(F)$. Prove that, if $AB = 0$, then $\text{rank } A + \text{rank } B \leq n$. (*Hint:* Think about the associated linear transformations L_A and L_B and use the dimension formula).

4. Consider the linear transformation $T_A : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $B \mapsto AB - BA$, where A is some 2×2 matrix. (*Note:* each T_A is a different linear transformation depending on A).

a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find the matrix of T_A with respect to

$$\beta = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

in terms of a, b, c, and d. (i.e., find $[T_A]_\beta^\beta$ in terms of a, b, c, and d).

b) Now, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find bases for $\ker T_A$ and $\operatorname{im} T_A$. Compute rank T_A and nullity T_A in this case.

c) *Extra Credit:* Which 2×2 matrices A commute with *all* 2×2 matrices B? (recall that two matrices M and N are said to *commute* if $MN = NM$). Justify your answer. (*Hint:* This has something to do with part (a) of this problem).

5. Consider the system of equations

$$\begin{aligned} 3x_1 + 2x_2 + x_3 - x_4 - x_5 &= 6 \\ x_1 - x_2 - x_3 - x_4 + 2x_5 &= 12 \\ -x_1 + 2x_2 + 3x_3 + x_4 - x_5 &= 18 \end{aligned}$$

Find all solutions using row reduction, and parameterize the solution space (as we did in class). What is the dimension of the solution space?

6. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, prove that if A is a 3×3 matrix of rank 1, then there exist a 3×1 matrix B and a 1×3 matrix C such that $BC = A$ (FIS p.158, #17).

7. Let $e_1 = (1, 2, 3, 4), e_2 = (2, 5, 10, 13), e_3 = (1, 4, 11, 18), e_4 = (0, 1, 4, 7)$.

a) Prove that $S = \operatorname{span}(\{e_1, e_2, e_3, e_4\})$ has dimension 3 (*Hint:* Row reduction).

b) Reduce the set $\{e_1, e_2, e_3, e_4\}$ to a basis for S (*Hint:* If you did the first part by row reduction, this step should be easy).

c) Can one select three elements e_i randomly and *a priori* claim that they are a basis for S? Why or why not?

8. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system always has a solution (FIS p.170, #10).