Problem Set 0: Techniques of Proof–Induction and Contradiction
Due Thursday, June 30th

Note: This homework is a bit longer than the "short" assignment will usually be. This has to do partly with the extra day, partly with the fact that much of this will be review to many people, and partly with the fact that these concepts are so essential if one is to be able even to *talk* about linear algebra, that they need to be mastered early on.

- **Induction**

Many statements in mathematics, particularly in linear algebra, are proved by *mathematical induction*. The framework is this: Let's say we have a statement that we want to prove for all the nonnegative integers $n \geq 0$, for instance, that the nth triangular number, or

$$\sum_{i=0}^{n} i,$$

is equal to $\frac{n(n+1)}{2}$. The idea is that in order to prove this statement for *all* nonnegative integers n, we only need to prove it for the *first* nonnegative integer (i.e., 0), and show that if we assume it is true for a given nonnegative integer n, then it is also true for $n+1$. This works because, if we go through this process, we will have shown that the statement is true for $n = 0$, and we will have also shown that this implies that the statement is true for $n = 1$, which in turn implies the statement for $n = 2$, and so on down the line.

In our particular example, the $n = 0$ case (or *base case*) is easy, as we just look at both sides of the equation and see that $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$. To do the induction step, let's assume that the statement is true for a given n, i.e., that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Then, to prove the statement for $n+1$, we write

$$\sum_{i=0}^{n+1} i = \left(\sum_{i=0}^{n} i\right) + (n+1).$$

But by the induction hypothesis (i.e., what we assumed was true), we know that the first term on the right hand side is equal to $\frac{n(n+1)}{2}$. So we obtain that

$$\sum_{i=0}^{n+1} i = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2},$$

which is exactly what we wanted to prove.

Question 1: Prove by induction that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Question 2: Prove by induction that, for all integers $k \geq 0$, $5^k - 4k - 1$ is divisible by

8.

Occasionally, one needs to use a variant of induction called *complete induction* (or *strong induction*). This is just like regular induction, except that, at the induction step, instead of assuming that the statement is true just for a given n, we assume that it is true for all nonnegative integers up to and including the given n. This is logically equivalent to standard induction (try to convince yourself of this!). Also note that everything I have said thus far is valid with any integer as the "base case," not just zero (so we can prove a statement for all integers greater than or equal to b by using b as the base case).

Question 3: Say that a bunny rabbit is hopping up a flight of stairs of length n. It hops either 2 steps at a time or 1 step at a time. For instance, if $n = 3$, then there are 3 different ways for the bunny to climb the stairs (it could hop 2 steps and then 1 step, 1 step and then 2 steps, or 1 step, then 1 step, then 1 step). Prove, using complete induction, that the number of different ways for the bunny to hop up the stairs is equal to F_n, where F_n is the nth Fibonacci number (recall that the Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and, for $n > 1$, $F_n = F_{n-1} + F_{n-2}$). Note: for a 0 length stairway, we say that there is 1 way to climb it. *Hint:* Think about how many ways there are for the bunny to make it to the point just before its last hop.

- **Logic**

Conditional statements of the form "If A, then B" abound in all areas of mathematics (and in life). We call A the *antecedent* and B the *consequent*. For instance, one might make the statement, "If an integer $n > 2$ is even, it is a composite number." This is clearly a true statement, as any time the antecedent holds, so does the consequent. However, one might also make the statement, "If an integer $n > 2$ is -5, it is a composite number." In mathematics, *this statement is also true!* Indeed, any time the antecedent holds (which is never), so does the consequent. The only way such a conditional statement can be false if it is possible for the antecedent to hold, but for the consequent to be false.

The *contrapositive* of a statement of the form "If A is true, then B is true," is the statement "If B is false, then A is false." We sometimes shorten the first statement and write "If A, then B," while the second statement becomes "If not B, then not A." The second statement is logically equivalent to the first statement in that one statement is true exactly when the other is true. (for instance "If it is sunny, then I will go outside" is logically equivalent to "If I do not go outside, then it is not sunny"). The *converse* of a statement of the form "If A, then B" is the statement "If B, then A." The converse of a statement is NOT logically equivalent to the given statement (for instance "If it is sunny, then I will go outside" is not logically equivalent to "If I go outside, it is sunny," because in the first case, I might go outside anyway, even if it is rainy).

Question 4: Write down the contrapositive of the statement, "Given a differentiable, real-valued function defined on $\mathbb{R}$, if it is non-constant, then its derivative is not uniformly zero."

Since contrapositives involve negation, one needs to know how to negate mathematical statements. For complicated statements involving lots of quantifiers, this can be a subtle matter. The two rules to keep in mind are

- The negation of the statement "For all A, we have B," is "For some A, we do not have B," or "There exists an A for which we do not have B," and vice versa.
- The negation of the statement "There exists an A for which we have B," is "For no A do we have B," or "There does not exist an A for which we have B," and vice versa.

These negations are correctly defined because, whenever one of the statements is true, its negation is false, and vice versa. Sometimes, you may have to do a bit of thinking in order to get a statement into one of these forms.

Question 5: Write down the negation of the statement "If a real square matrix has nonzero determinant, it is invertible over the real numbers." (Extra credit: which is true, the statement or the negation?)

Question 6: Write down the contrapositive of the same statement (there are several ways to do this).

Sometimes, it is easier to prove a mathematical statement by *contradiction*. For example, let's say we are asked to prove that, given a polynomial $f(x)$ with real coefficients, and a real number a such that $f(a) \neq 0$, then $x - a$ does not divide $f(x)$. To prove this by contradiction, we assume that what we want to prove is false, i.e., that $x - a$ does divide $f(x)$. In this case, we have $f(x) = (x - a)g(x)$ for some polynomial g. But then $f(a) = (a - a)g(a) = 0$, which contradicts the assumption that $f(a) \neq 0$. Thus, what we want to prove must in fact be true. (Really, what we are proving is the contrapositive of the statement "If $f(a) \neq 0$, then $x - a$ does not divide $f(x)$.")

Question 7: The *pigeonhole principle* says that if one has n pigeons and k pigeonholes ($k \neq 0$), with $n \geq pk + 1$ for some given $p \geq 0$, then any way of stuffing the pigeons into the holes will result in at least one hole containing more than p pigeons. Prove this by contradiction.